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SHEAR FORCES IN THE CONNECTION OF STRUCTURAL ELEMENTS UNDER BENDING

CZESŁAW MACHELSKI

Bridge Department, Institute of Civil Engineering, Wrocław University of Technology
Wybrzeże Wyspiańskiego 27, 50-370 Wrocław, Poland. E-mail:czeslaw.machelski@pwr.wroc.pl

Abstract: A beam consisting of elements connected flexibly along their axis is discussed in the paper. An example of a structural system, constructed with the use of different materials is a composite girder consisting of a steel beam connected with a concrete slab. Full interaction of elements connected discontinuously (discretly) along the axis of the beam is a particular (theoretical) case. Taking into account functions of shear forces and stiffness k, a connection, can be classified as rigid when $k > 1400$ MN/m^2. In this case, the results obtained from the classic beam (homogeneous) model are sufficient for design purposes. In the cases presented in the paper, the results obtained using classic model may considerably deviate from the results obtained using layered model, considered to be the accurate one. The examples of analyses given in the paper show the possibility for the use of Fourier series in calculating shear forces in the case of discrete connections. Using this solution, a compatibility of shear forces with values calculated on the basis of design solution was proved to be valid only in the case of full interaction (no shear slip). As the connection flexibility increases, the discrepancy between the functions of shear forces along the length of a girder rises. Regular arrangements of shear connectors were considered in the paper. The function of connection stiffness $k(x)$ can have a significant influence on the distribution of shear force $t(x)$.

1. INTRODUCTION

A structure consisting of elements connected flexibly along their axis is considered in the paper. An example of such a system, consisting of elements made of different materials, is a composite steel-concrete girder, as in Fig. 1. Full composite beam-slab interaction is a theoretical case, assuming non-deformable connection between the components of the girder. The steel-concrete connection is usually flexible and therefore the distribution of unit deformations along the height of the cross-section is discontinuous, as in Fig. 1. Typical shear studs, which are common in bridge structures [1]–[4] can be described as flexible connectors, because the stiffness of connection does not prevent the shear slip in the steel-concrete interface zone. This results in redistribution of internal forces in the girder and in reduction of its flexural stiffness.

An example of a homogeneous structural system, which is considered in detail below, are two steel corrugated plates, presented in Fig. 2. This configuration, consisting of two steel plates with identical cross-section, is commonly used for soil–steel corrugated plate structures [5]–[7]. The plates are connected longitudinally using bolts, as in Fig. 2, and transversally by overlapping and using the same type of bolts. The assumed example of corrugated plates, used in soil-steel structures is characterized by a rather small stiffness of bolt connection. Such a type of connection was researched in [6], [7] in full scale tests of bending a plate, simply supported at edges parallel to corrugation, to a cylindrical surface.

In both cases analyzed (shown in Fig. 1 and Fig. 2) the global bending moment M distributes into resultant internal forces M_b, N_b, M_n, N_n. This is convenient in calculations and in the case given in Fig. 1 enables modeling properties of concrete (e.g., creep). The possibility of using Bernoulli's principle of flat sections in separate parts of the structure is important in this case.

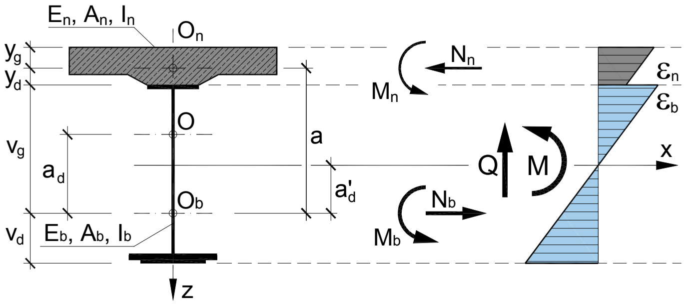


Fig. 1. Composite girder – cross-section and internal forces

2. CONNECTION STIFFNESS

The connection can be described by its load bearing capacity and stiffness, determined in the push-out tests as

$$k = \frac{t}{\delta} \tag{1}$$

where

t – shear force in the connection,

δ – mutual displacement of elements along the contact zone (interface).

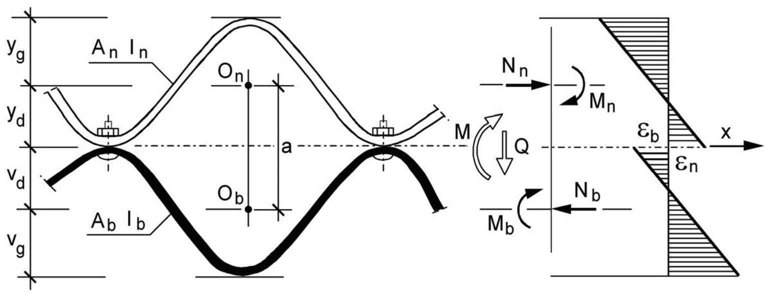


Fig. 2. Internal forces in the circumferential section of a corrugated steel plate

The value of k from (1) depends in general on the type of shear connectors (studs, rivets, screws, bolts) and on the physical characteristics of structural materials (steel/steel, steel/concrete, concrete/wood). The connection stiffness is affected by a load type: short-term, cyclic (periodic), long-term (including the rheological processes). Load intensity has also an essential influence. The value of k may vary in the same structure and with the same type of connection due to the different load type. For these reasons the value of k cannot be unequivocally defined if the load characteristic is not given.

The crucial problem which occurs during calculation of internal forces is the method of taking into account the stiffness of the connection. In the paper, the focus is put on the analysis of shear forces in the connection, as the result of bending of the beam, depending on the stiffness of the connection. For the evaluation of the proposed solution, comparative algorithms derived from two girder models: beam (classic, homogeneous) and composite (treated as accurate one) have been applied.

3. FUNCTIONS OF SHEAR FORCES IN THE CONNECTION OF GIRDER'S STRUCTURAL ELEMENTS

To determine the shear force in a girder, a beam model is commonly used and the following equation is applied

$$t(x) = \frac{S}{I} Q(x) \tag{2}$$

where

- S – static moment of area, calculated relative to the neutral axis of the section,
- I – moment of inertia, calculated relative to the neutral axis of the section,
- $Q(x)$ – transverse force function along the girder length.

Force $t(x)$ is distributed along the interface of connected elements, as in Fig. 3.

Formula (2) is obtained from the equation of shear stress in the homogeneous cross-section [6], [7] or in the structure consisting of fully interacting elements (e.g., connected by welding)

$$\tau = \frac{Q \cdot S}{I \cdot b}\,. \tag{3}$$

As results from formula (2), when both joined elements are prismatic, the function of shear force is proportional to the diagram of $Q(x)$. Therefore, it can be concluded that along the sections of a girder where the transverse force does not exist $Q(x)$, the shear force $t(x)$ does not appear either. This conclusion, resulting from formula (2), is not general. It can be impaired after the consideration of few simple static and load schemes of beams.

In double-layered structure, shown in Fig. 1, Fig. 2 and Fig. 3, a base (reference) element is distinguished. This results from its presence in the solution of the problem given below. Internal forces in the secondary element are derived from the forces in the base element.

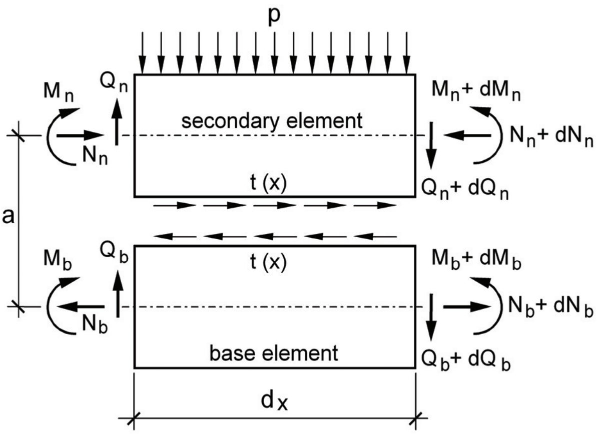


Fig. 3. Internal forces in the section of a layered (composite) element

Using the layered (composite) model as in Fig. 3, one obtains the general relationship of the axial force in the beam N_b dependent on the resultant bending moment M in the cross-section analyzed [1]

$$\frac{EI_b}{a \cdot k}(\kappa_0 - \mu_0)\left(\frac{d^2 N_b}{dx^2} - \frac{1}{k}\frac{dN_b}{dx}\frac{dk}{dx}\right) - \frac{\kappa_0}{\mu_0} a N_b + M = 0 \tag{4}$$

after introducing the geometric characteristics of cross-section

$$\mu_0 = \frac{a^2 A_n A_b}{(A_n + A_b) I_b}, \tag{5}$$

$$\kappa_0 = 1 + \mu_0 + \frac{I_n}{I_b} = \mu_0 + \frac{I_b + I_n}{I_b}. \tag{6}$$

In equations (4)–(6) the designations of geometrical and physical characteristics of the girder (Fig. 1 and Fig. 2) in the form of indices: n (top element) and b (bottom element) are applied. Obviously when the structural system consists of two different materials, the geometrical characteristics of one of them (e.g. concrete slab) have to be adjusted. For this reason, the values A_n and I_n, given in the equations, have to be treated as reduced to the material of the beam on the basis of the proportion E_b/E_n [1]. The distance a between the centroids of both elements is constant.

In the case of a composite girder given in Fig. 1, the value of I_n/I_b is small and can usually be omitted

$$\kappa_0 \approx 1 + \mu_0 \tag{7}$$

and when the system is composed of equivalent elements ($I_n = I_b$), as in Fig. 2, one obtains

$$\kappa_0 = 2 + \mu_0 . \tag{8}$$

By using the layered (composite) model of the girder and equation (4), on the basis of the function of bending moment $M(x)$ one obtains the axial force $N_b(x)$. Remaining internal forces, given in Fig. 2 and Fig. 1, are calculated using the following conditions which are taken into consideration in equation (4):

- equilibrium of axial forces in the section

$$N_n = N_b , \tag{9}$$

- compatibility of curvatures

$$M_n = \frac{I_n}{I_b} M_b , \tag{10}$$

- equilibrium of bending moments in the section

$$M = M_b + M_n + aN_b , \tag{11}$$

hence

$$M_b = \frac{M - aN_b}{\kappa_0 - \mu_0} . \tag{12}$$

In the transformation of (11) to the form of (12), the relationships defined in (10) and (6) were used in the following equation

$$M_b + M_n = M_b + \frac{I_n}{I_b} M_b = M_b \left(\frac{I_b + I_n}{I_b} \right) = M_b (\kappa_0 - \mu_0) . \tag{13}$$

From the equation of cross-section area static moments of composite girder's elements, given in the form

$$A_n (a - a_d) = A_b \cdot a_d , \tag{14}$$

the following relationship is obtained

$$A_n \cdot a = A_b \cdot a_d + A_n \cdot a_d = (A_b + A_n) a_d , \tag{15}$$

and further the position of the neutral axis, i.e., the point O relative to O_b, as in the formula

$$a_d = \frac{A_n}{A_b + A_n} a \, . \tag{16}$$

Obviously, in the case of flexible connection the neutral axis takes the position a'_d . Substituting a_d into formula (5), one can define

$$\mu_0 = \frac{a a_d A_b}{I_b} \, . \tag{17}$$

The factor $a_d A_b$ in equation (17) is the static moment of cross-section area relative to the neutral axis

$$S = a_d A_b, \tag{18}$$

which appears in (2) and (3). Therefore, from formulas (17) and (18) one can write the relationship using the cross-section characteristic μ_0 from (4)

$$I_b \mu_0 = a \cdot S \, . \tag{19}$$

Geometric characteristic of the cross-section μ_0 is also the component of Steiner's formula, as in the following equation

$$I = I_b + I_n + a_d^2 A_b + (a - a_d)^2 A_n = I_b + I_n + a a_d A_b = (1 + \mu_0) I_b + I_n \, . \tag{20}$$

Comparison of (6) and (20) shows that κ_0 can be used for determining the girder's cross-section moment of inertia

$$I = \kappa_0 I_b. \tag{21}$$

Substituting formulas (19) and (21) into (2) one obtains the relationship between the shear force and transverse force $Q(x)$, when the previously derived geometric characteristics of the cross-section are used

$$t(x) = \frac{\mu_0 \cdot I_b}{a \cdot \kappa_0 \cdot I_b} Q(x) = \frac{\mu_0}{a \cdot \kappa_0} Q(x) \, . \tag{22}$$

Directly from (22) the following relationship is obtained

$$T_q(x) = a \cdot t(x) = \frac{\mu_0}{\kappa_0} Q(x) \, . \tag{23}$$

Obviously, it is derived from (2), using relationship (18)

$$T_q(x) = a \cdot t(x) = \frac{S \cdot a}{I} Q(x) = \frac{a a_d A_b}{I} Q(x) \, . \tag{24}$$

Due to the cross-sectional configuration, relationships between $T_q(x)$ and $Q(x)$ take form given in (24a) and (24b), due to the value of I_n related to I in (20) and to the parameter κ_0.

In the case of a composite girder, given in Fig. 1, using the relationship (7) one gets

$$T_q(x) \approx \frac{\mu_0}{1+\mu_0} Q(x), \tag{24a}$$

while during analysis of corrugated shells, given in Fig. 2, one obtains from (8)

$$T_q(x) = \frac{\mu_0}{2+\mu_0} Q(x). \tag{24b}$$

In the model of a homogeneous girder, described by equation (2), one uses the geometrical characteristic μ_0, which is applied in the composite model.

In the case of the layered (composite) model of a girder (4), if $k \to \infty$ the relationship (4) simplifies to the form

$$\frac{\kappa_0}{\mu_0} a N_b = M . \tag{25}$$

After differentiating of (25) with respect to x and taking into account the following relationship

$$t(x) = \frac{dN_b}{dx} , \tag{26}$$

one obtains

$$\frac{\kappa_0}{\mu_0} a \frac{dN_b}{dx} = \frac{\kappa_0}{\mu_0} a t(x) = \frac{dM}{dx} = Q(x) . \tag{27}$$

From the comparison of formulas (27) and (22) the full compatibility of calculated shear forces $t(x)$ can be seen, though completely different calculation models have been applied. This conclusion concerns the situation when full interaction of girder's elements takes place, as in (25). The characteristics μ_0 and κ_0 in (27) are also the reason for taking in (4) the parameters of layered (composite) model, given in (5) and (6).

From the comparison of equations (2), (26) and (4) it is evident that in order to calculate the shear forces, other types of internal forces, i.e., $Q(x)$ or $M(x)$, can be used. Therefore, the functions of shear force $t(x)$ in the same and identically connected elements, which are similarly loaded but defined using completely different models, may vary.

4. INFLUENCE OF CONNECTION STIFFNESS ON SHEAR FORCE

As an example constant connection stiffness along the length of the element (i.e., k = const, $dk/dx = 0$) is considered, as in the case of regular arrangement of connectors of the same type. This assumption simplifies equation (4) to the following form

$$\frac{EI_b}{a \cdot k}(\kappa_0 - \mu_0)\frac{d^2 N_b}{dx^2} - \frac{\kappa_0}{\mu_0} a N_b + M = 0\,. \tag{28}$$

Examples [3] of shear force functions $t(x)$ in a simply supported beam, loaded by two forces $P = 1$ kN, as in Fig. 4, when $L = 4.830$ m and $b = d = L/3$ are presented below. The geometrical characteristics of connected elements (corrugated plates), as in Fig. 2, considered in the analysis are $A_b = A_n = 9.81$ mm^2/mm, $a = 150.65$ mm, $I_b = I_n = 24165$ mm^4/mm [3], [4]. Taking into account this data the geometric characteristic of the cross-section from (5) was calculated as

$$\mu_0 = \frac{a^2 A_b}{2 I_b} \tag{29}$$

and is equal to the value $\mu_0 = 4.6067$ and from (8) $\kappa_0 = 6.6067$. Denotation of sections along the beam is given in Fig. 4. This load scheme has been assumed as in the load testing of layered plates [3], for $L = 4.83$ m, $b = d = L/3$, under cylindrical bending.

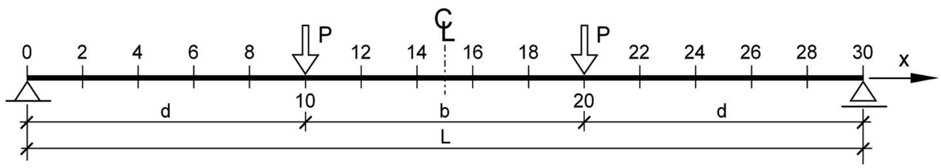


Fig. 4. Static scheme of the beam

The results of calculations obtained on the assumption of the constant value of k are given Fig. 5. Functions $T(x)$ with variable parameter C (and thus k) are calculated using relationships (28) and (26) and formula

$$T(x) = a \cdot t(x)\,. \tag{30}$$

The stiffness of the connection k is related to the modulus of elasticity of the beam's material $E = 205000$ MN/m^2, which occurs in (4) and (28) as the dimensionless parameter

$$c = \sqrt[4]{\frac{E}{k}}\,. \tag{31}$$

Due to the large range of E/k values, parameter c defined in (31) has been proposed. It enables us to obtain a clearer form of diagrams given in Fig. 5 and Fig. 6.

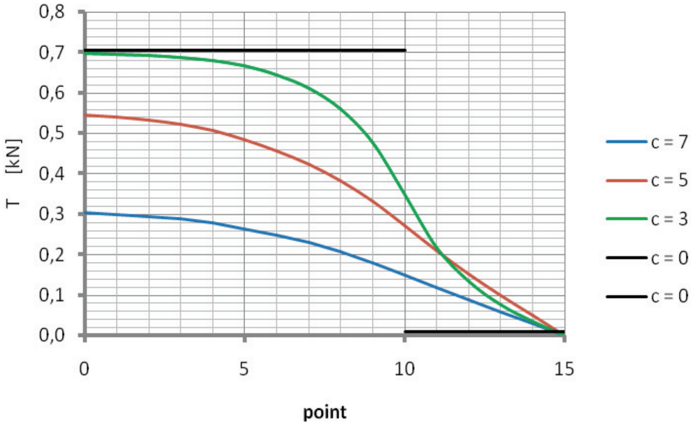


Fig. 5. The function of shear force in the case of constant stiffness k

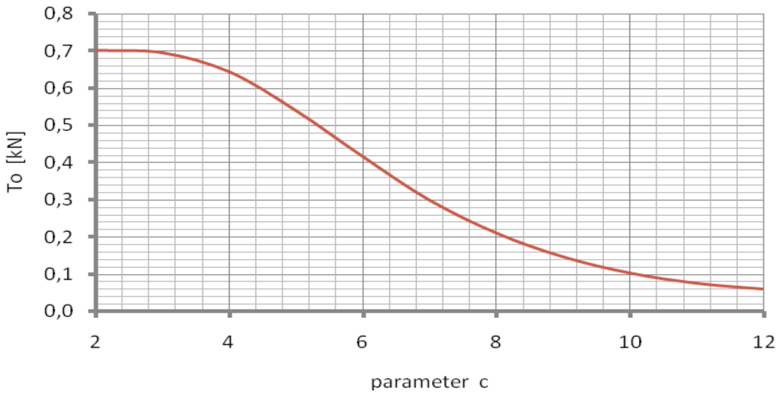


Fig. 6. Maximum value of shear force versus the connection stiffness

In the case of applying formula (2) the transverse force $Q(x)$ takes the constant value: $Q = P = 1$ kN in the range of $0 < x < d$, while $Q = 0$ when $d < x < d + b$. From formula (24b) one obtains in the first range of x the value of shear force

$$T_q = \frac{\mu_0}{2+\mu_0} P = \frac{4.6067}{6.6067} 1.0 = 0.69728 \text{ kN}$$

and $T_q = 0$ in the zone between forces P. This graph is denoted as $c = 0$ (full interaction). All graphs of $T(x)$ obtained from the layered (composite) model when $c > 3$ are similar and in every case they significantly deviate from the shape of $Q(x)$. With decrement of the value of k (i.e., with increase of the value of c) the similarity to the graph of $T(c = 0)$, which is determined on the basis of $Q(x)$, fades.

In the case of geometrical characteristics of cross-section other than given in Fig. 2, different values of μ_0 from (5) and κ_0 from (6) and obviously different value of T_q are obtained.

Maximum values of shear force over the support (point 0, as in Fig. 4), denoted by To, are given in Fig. 6. This graph has been made on the basis of the solutions in Fig. 5. The variable in the analysis is the connection stiffness k, referred to the parameter c (30). From the shape of graph (Fig. 6) it can be noticed that if $c < 3.5$ (i.e., $k > 20500/3.5^4 = 1366$ MN/m^2), the results of calculations are similar to those obtained from equation (24). When $c \to 0$ (or $k \to \infty$) one obtains the particular solution given in (25). Although the values of k are small, shear forces are repeatedly smaller than T, calculated from (31).

5. APPLICATION OF FOURIER SERIES

Constant connection stiffness along the length of the element (k = const), as in equation (28), is considered below. The shapes of diagram given in Fig. 5 suggest usefulness of the Fourier series [9] for determining shear forces in the joint of connected elements. The loads given in Table 1, transformed into the Fourier series, for arbitrary harmonic m, take the general form

$$p(x) = \sum_m p_m \sin(a_m x)\,, \tag{32}$$

where

$$a_m = \frac{m\pi}{L}\,. \tag{33}$$

The function of displacement can be written similarly to the loads

$$w(x) = \sum_m \; w_m \sin(a_m x)\;. \tag{34}$$

From the relationship of deflections and loads of a beam with the bending stiffness EI, as in the formula

$$EI \cdot \frac{d^4 w}{dx^4} = p \tag{35}$$

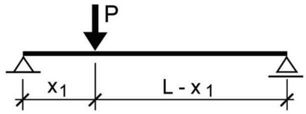
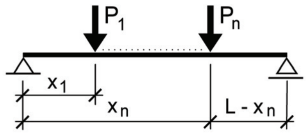
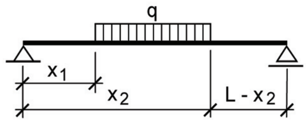
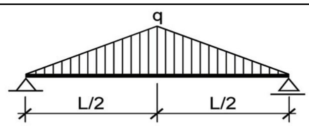
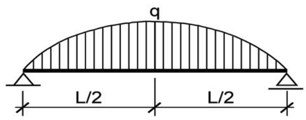
one can obtain the following relationship for the arbitrary harmonic m

$$EI \cdot a_m^4 \cdot w_m \sin(a_m x) = p_m \sin(a_m x) \tag{36}$$

and from the above expression

$$w_m = \frac{p_m}{EI} a_m^{-4}. \tag{37}$$

Table 1. The function of transforming load into the Fourier series

No.	Load scheme	Function p_m
1		$p_m = \frac{2P}{L}\sin(a_m x_1)$
2		$p_m = \frac{2}{L}\sum_{i=1}^{n} P_i \sin(a_m x_i)$
3		$p_m = \frac{2q}{m\pi}(\cos(a_m x_1) - \cos(a_m x_1))$
4		$p_m = \frac{8q}{(m\pi)^2}\sin(m\pi/2)$
5		$p_m = \frac{32q}{(m\pi)^3}\sin^2(m\pi/2)$

From the relationship of deflections and bending moments

$$M = EI \cdot \frac{d^2 w}{dx^2} \tag{38}$$

one obtains for the arbitrary harmonic m

$$M_m = p_m a_m^{-2}. \tag{39}$$

Taking it as in the case of (32) and (34), the function of axial force, in the element "b", as in Figs. 1–3,

$$N_b = \sum_m N_m \sin(a_m x) \tag{40}$$

one obtains for arbitrary *m*

$$\frac{d^2 N_b}{dx^2} = -a_m^2 N_m \sin(a_m x) . \tag{41}$$

After substituting (39), (40) and (41) into formula (28), one gets for arbitrary harmonic *m*

$$-\left[\frac{EJ_b}{ak}(\kappa_0 - \mu_0)a_m^2 + \frac{a\kappa_0}{\mu_0}\right] N_m \sin(a_m x) + p_m a_m^{-2} \sin(a_m x) = 0 \tag{42}$$

and after transformation

$$N_m\left[\frac{EI_b}{ak}(\kappa_0 - \mu_0)a_m^2 + \frac{a\kappa_0}{\mu_0}\right] = p_m a_m^{-2} . \tag{43}$$

Application of the relationship (26) allowed the function of shear force to be obtained

$$t = \frac{dN_b}{dx} = -\sum_m N_m a_m \cos(a_m x) \tag{44}$$

and after substituting (43) the final form of the equation is developed

$$t(x) = \sum_m \frac{\left(\frac{p_m}{a_m a}\right)\cos(a_m x)}{\frac{EI_b}{k}\left(\frac{a_m}{a}\right)^2 (\kappa_0 - \mu_0) + \frac{\kappa_0}{\mu_0}} . \tag{45}$$

In formula (45), the factor which determines the axial force for the whole structure can be distinguished in terms of the Fourier series

$$Q(x) = \sum_m \frac{p_m}{a_m} \cos(a_m x) . \tag{46}$$

Using the notation applied in formula (24), one can derive the following formula for the shear force

$$T(x) = a \cdot t(x) = \sum_m \frac{\left(\frac{p_m}{a_m}\right)\cos(a_m x)}{\frac{EI_b}{k}\left(\frac{a_m}{a}\right)^2 (\kappa_0 - \mu_0) + \frac{\kappa_0}{\mu_0}} . \tag{47}$$

In the case of full interaction, i.e., when $k \to \infty$, the following relationship can be obtained from (47)

$$T(x) = t(x) \cdot a = \frac{\mu_0}{\kappa_0} \sum_m \frac{p_m}{a_m} \cos(a_m x) = \frac{\mu_0}{\kappa_0} Q(x) \tag{48}$$

compatible with the one calculated in (27).

6. INFLUENCE OF LOADING SCHEME ON SHEAR FORCE

Below are presented the results of analysis considering the corrugated plate shown in Fig. 2 [5]–[7] and the load scheme given in Fig. 4. Geometric characteristics are: a = 150.65 mm, μ_0 = 4.6067 and κ_0 = 6.6067. As the loading of the beam there are considered the forces given in Table 1. The intensity of the load is selected to ensure equal reactions R (and vertical forces Qo) on both supports in every scheme:

- scheme 2 (Fig. 4) $P = 1$ kN,
- scheme 3 $x_1 = L - x_2 = L/3$ $R = \frac{qL}{6} = 1$ kN $\to q = 6/L$ [kN/m],
- scheme 4 $R = \frac{qL}{4} = 1$ kN $\to q = 4/L$ [kN/m],
- scheme 5 $R = \frac{qL}{3} = 1$ kN $\to q = 3/L$ [kN/m].

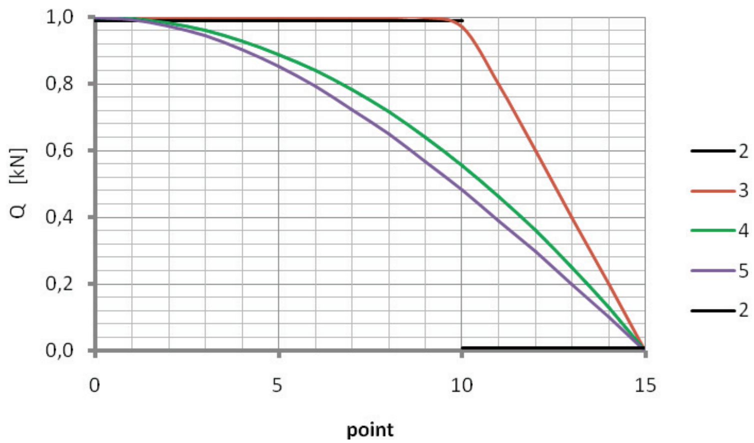


Fig. 7. The diagrams of transverse force $Q(x)$ along the beam

Figure 7 presents the diagrams of transverse force $Q(x)$, obtained on the basis of formula (46). In the case of full interaction, equation (48) can be used for the calculation of shear force T. Then we get $\mu_0/\kappa_0 = 0.69728$, as in formula (30). Two lines in scheme 2 result from the function of vertical force, i.e., two different values at point 10 – on the left side $Q_{10} = 1.0$ and on the right side $Q_{10} = 0$, as in Fig. 4. Figure 8 presents the graphs of T calculated according to (47), assuming $k = 328$ MN/m^2 and thus $E/k = 625$ and $c = 5$. Despite significant differences in load schemes, similar shapes of functions $T(x)$ have been obtained. The utmost differences in the shape of graphs $Q(x)$ and $T(x)$ appear in load schemes 2 and 3.

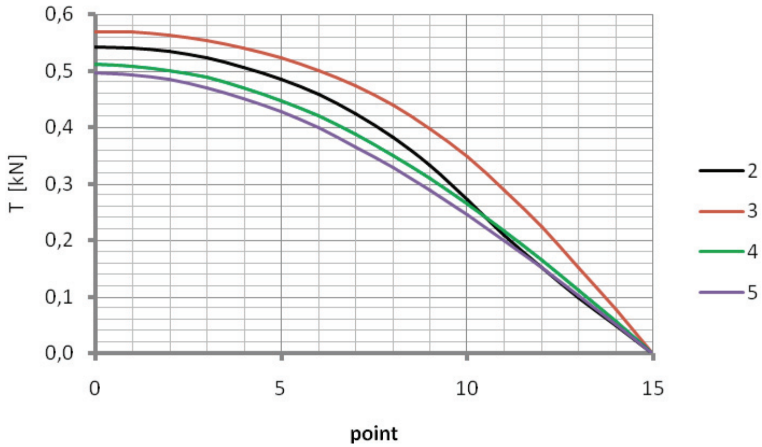


Fig. 8. The diagrams of shear forces $T(x)$ along the beam

7. SUMMARY

Perfect connection and full interaction of elements, connected discretly in a composite structure is a particular (theoretical) case. According to the function of shearing force, connections can be classified as rigid, when the stiffness $k > 1400$ MN/m^2, for which the results obtained from the classic beam (homogeneous) model are sufficient for design purposes. In the cases presented in the paper, the results obtained from equation (2) usually deviate from the results obtained with the use of composite model (4), which is considered to be the accurate one. Analyses given in the paper show the possibility of applying formula (47) in design of connections of elements. The results show the compatibility of shear forces in the connection only in the case of full interaction (no shear slip). Along with the increase of the connection flexibility the discrepancy in the functions of shear force along the girder also increases.

Regular arrangement of shear connectors (constant stiffness) was considered in the paper. The function of the connection stiffness $k(x)$ can have a significant influence on the distribution of shear force $t(x)$ [7]. Results of analyses can be useful in designing connections with the use of flexible connectors, as in examples given in the paper.

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