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ON THE STABILITY OF TWO STRATIFIED WALTERS B' VISCOELASTIC SUPERPOSED FLUIDS

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Abstract: The stability of the plane interface separating two Walters B' viscoelastic fluids has been studied. The density, viscosity and viscoelasticity are assumed to be exponentially varying. For stable stratifications, the system is found to be stable or unstable under certain conditions. The system is found to be unstable for unstable stratifications. Also the growth rates are found to decrease as well as increase with the increase in kinematic viscosity and kinematic viscoelasticity. The effect of a variable horizontal magnetic field is also considered. For the stable stratifications, in the hydromagnetic case also, the system is found to be stable or unstable under certain conditions. However, for the unstable stratifications, the magnetic field has got stabilizing effects.

1. INTRODUCTION

When two fluids of different densities are superposed one over the other (or accelerated towards each other), the instability of the plane interface between the two fluids, when it occurs, is called the Rayleigh–Taylor instability. In general, it is derived from the character of the equilibrium of an incompressible stratified heterogeneous fluid. HIDE [1] has treated the character of the equilibrium of a viscous, incompressible, rotating fluid of variable density and found that rotation stabilizes the potentially unstable arrangement for certain wave number range. He has considered the directions of angular velocity vector and gravity vector (in the direction of the vertical) to be inclined. In another study, HIDE [2] has studied the case of a viscous, incompressible, electrically conducting fluid of variable density in the presence of a vertical magnetic field and found that magnetic field considerably stabilizes the configuration and it is possible to have oscillatory motions in the presence of magnetic field even if the configuration is thoroughly unstable (density wise). CHANDRASEKHAR [3] has given a detailed account of the instability of the plane interface between two incompressible and viscous fluids of different densities when the lighter is accelerated into the heavier.

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The influence of viscosity on the stability of the plane interface separating two incompressible superposed fluids of uniform densities, when the whole system is immersed in a uniform horizontal magnetic field, has been studied by BHATIA [4]. He has carried out the stability analysis for two fluids of equal kinematic viscosities and different uniform densities. SHARMA [5] has studied the stability of Oldroydian viscoelastic superposed conducting fluids in the presence of a uniform magnetic field. Generally, the magnetic field has a stabilizing effect on the instability but there are a few exceptions also. For example, KENT [6] has studied the effect of a horizontal magnetic field which varies in the vertical direction on the stability of parallel flows and has shown that the system is unstable under certain conditions, while in the absence of magnetic field the system is known to be stable. SHARMA and SHARMA [7] have studied the stability of the plane interface separating two viscoelastic (Oldroyd) superposed fluids of uniform densities. FREDRICKSEN [8] has given a good review of non-Newtonian fluids, whereas JOSEPH [9] has also considered the stability of viscoelastic fluids. Molten plastics, petroleum oil additives and whipped cream are the examples of incompressible viscoelastic fluids.

The fluids have been considered to be Newtonian or viscoelastic (Maxwellian or Oldroydian) in all the above studies. There are many elastico-viscous fluids that cannot be characterized by Maxwell's constitutive relations or OLDROYD'S [10] constitutive relations. One such class of viscoelastic fluids is Walters B' viscoelastic fluid. CHAKRABORTY and SENGUPTA [11] have studied the flow of unsteady viscoelastic (Walters B' liquid) conducting fluid through two porous concentric non-conducting infinite circular cylinders rotating with different angular velocities in the presence of uniform axial magnetic field. In another study, SHARMA and KUMAR [12] have studied the steady flow and heat transfer of Walters fluid (Model B') through a porous pipe of uniform circular cross-section with small suction. The Rayleigh–Taylor instability of two superposed electrically conducting Walters B' elastico-viscous fluids in hydromagnetics has been considered by SHARMA and KUMAR [13].

Keeping in mind the importance of non-Newtonian fluids in geophysical fluid dynamics, chemical technology and petroleum industry, the stability of stratified Walters B' viscoelastic fluid is considered. The stability of stratified Walters B' viscoelastic fluid is also considered when the fluid is electrically conducting and a variable horizontal magnetic field pervades the system.

2. MATHEMATICAL FORMULATION OF THE PROBLEM AND PERTURBATION EQUATIONS

Here we consider a static state, in which an incompressible Walters B' elastico-viscous fluid of the depth d is arranged in horizontal strata. The pressure p, density ρ, viscosity μ and viscoelasticity μ' are the functions of the vertical z-coordinate only.

The character of the equilibrium of this initial static state is determined by supposing that the system is slightly disturbed and then by following its further evolution.

Let T_{ij}, τ_{ij}, e_{ij}, δ_{ij}, v, and x denote the stress tensor, shear stress tensor, rate-of-strain tensor, Kronecker delta, velocity vector and position vector, respectively. The constitutive relations for the Walters B' viscoelastic fluid are

$$T_{ij} = -p\delta_{ij} + \tau_{ij},$$

$$\tau_{ij} = 2\left[\mu - \mu' \frac{\partial}{\partial t}\right] e_{ij}, \tag{1}$$

$$e_{ij} = \frac{1}{2}\left[\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right].$$

Then the momentum balance and mass balance equations for Walters B' incompressible viscoelastic fluid are

$$\rho\left[\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v}\right] = -\nabla p + \rho\vec{g} + \rho\left(\upsilon - \upsilon'\frac{\partial}{\partial t}\right)\nabla^2\vec{v} + \left[\frac{d\mu}{dz} - \frac{\partial}{\partial t}\frac{d\mu'}{dz}\right]\left(\frac{\partial w}{\partial \overline{x}} + \frac{\partial \vec{v}}{\partial z}\right), \tag{2}$$

$$\nabla\cdot\vec{v} = 0. \tag{3}$$

Here $\vec{v}(u,v,w)$, ρ and p, respectively, denote the velocity of fluid, the density and the pressure. $\mu, \mu', \upsilon(=\mu/\rho)$ and $\upsilon'(=\mu'/\rho)$ stand for viscosity, viscoelasticity, kinematic viscosity and kinematic viscoelasticity, respectively, $\vec{g} = (0,0,-g)$ is the acceleration due to gravity and $\overline{x} = (x,y,z)$.

Since the density of a fluid particle remains unchanged as we follow it with its motion, we have

$$\frac{\partial\rho}{\partial t} + (\vec{v}\cdot\nabla)\rho = 0. \tag{4}$$

This additional equation (for the validity of mass balance equation) needs to be satisfied as the fluid is heterogeneous. For homogeneous fluid, this is identically satisfied.

Let $\vec{v}(u, v, w)$, $\delta\rho$ and δp denote the perturbations in velocity (0, 0, 0), density ρ and pressure p, respectively. Then the linearized perturbation equations appropriate to the problem are

$$\rho\frac{\partial \vec{v}}{\partial t} = -\nabla\delta p + \vec{g}\delta\rho + \rho\left(\upsilon - \upsilon'\frac{\partial}{\partial t}\right)\nabla^2\vec{v} + \left[\frac{d\mu}{dz} - \frac{\partial}{\partial t}\frac{d\mu'}{dz}\right]\left(\frac{\partial w}{\partial \overline{x}} + \frac{\partial \vec{v}}{\partial z}\right), \tag{5}$$

$$\nabla \cdot \vec{v} = 0\,, \tag{6}$$

$$\frac{\partial}{\partial t}\delta\rho = -wD\rho\,, \tag{7}$$

where $D = d/dz$.

3. DISPERSION RELATION

Analyzing the disturbances to normal modes, we seek solutions whose dependence on x, y and t is given by

$$\exp\,(ik_x x + ik_y y + nt), \tag{8}$$

where n is the rate at which the system departs from the equilibrium, k_x, k_y are horizontal wave numbers along x- and y-directions, respectively, and $k[= (k_x^2 + k_y^2)^{1/2}]$ is the resultant wave number.

Using expression (8), equations (5)–(7) become

$$\rho n u = -ik_x \delta p + \rho(\upsilon - \upsilon' n)(D^2 - k^2)u + (ik_x w + Du)(D\mu - nD\mu')\,, \tag{9}$$

$$\rho n v = -ik_y \delta p + \rho(\upsilon - \upsilon' n)(D^2 - k^2)v + (ik_x w + Dv)(D\mu - nD\mu')\,, \tag{10}$$

$$\rho n w = -D\delta p + \rho(\upsilon - \upsilon' n)(D^2 - k^2)w - g\delta\rho + 2Dw(D\mu - nD\mu')\,, \tag{11}$$

$$ik_x u + ik_y v + Dw = 0\,, \tag{12}$$

$$n\delta\rho = -wD\rho\,. \tag{13}$$

Multiplying equations (9) and (10) by $-ik_x$ and $-ik_y$, respectively, adding the resultant equations and using (12) in it, we obtain

$$\rho n D w = -k^2 \delta p + \rho(\upsilon - \upsilon' n)(D^2 - k^2)Dw + (D\mu - nD\mu')(D^2 + k^2)w\,. \tag{14}$$

Eliminating δp between equations (11) and (14) and using (13) we obtain

$$n[D(\rho D w) - k^2 \rho w] - \{D[\rho(\upsilon - \upsilon' n)(D^2 - k^2)Dw]$$
$$- k^2 \rho(\upsilon - \upsilon' n)(D^2 - k^2)w\} + \frac{gk^2}{n}(Dp)w$$
$$- \{D[(D\mu - nD\mu')(D^2 + k^2)w] - 2k^2(D\mu - nD\mu')(Dw)\} = 0. \tag{15}$$

4. THE CASE OF EXPONENTIALLY VARYING DENSITY, VISCOSITY AND VISCOELASTICITY

Here we consider the stratifications in density, viscosity and viscoelasticity of the fluid of the depth d as

$$\rho = \rho_0 e^{\beta z}, \quad \mu = \mu_0 e^{\beta z}, \quad \mu' = \mu_0' e^{\beta z}, \tag{16}$$

where ρ_0, μ_0, μ_0' and β are constants and so the kinematic viscosity

$$\upsilon \left(= \frac{\mu}{\rho_0} = \frac{\mu_0}{\rho_0} \right),$$

the kinematic viscoelasticity

$$\upsilon' \left(= \frac{\mu'}{\rho_0} = \frac{\mu_0'}{\rho_0} \right)$$

are constant everywhere.

We consider the case of two free boundaries. Let us assume that $\beta d << 1$, i.e., the variation of density at two neighbouring points in the velocity field which is much less than the average density, has a negligible effect on the inertia of the fluid. The boundary conditions for the case of two free surfaces are

$$w = D^2 w = 0 \quad \text{at} \quad z = 0 \quad \text{and} \quad z = d. \tag{17}$$

The proper solution of equation (15) satisfying (17) is

$$w = w_0 \sin \frac{m\pi z}{d}, \tag{18}$$

where w_0 is a constant and m is an integer.

Using the stratifications of the form (16) and neglecting the effect of heterogeneity on the inertia, we obtain

$$(D^2 - k^2)w + \frac{g\beta k^2}{n^2} w - \frac{1}{n}(\upsilon_0 - n\upsilon_0')(D^2 - k^2)^2 w - \frac{\beta^2}{n}(\upsilon_0 - n\upsilon_0')(D^2 + k^2)^2 w = 0. \tag{19}$$

Substituting (18) in equation (19) and simplifying, we obtain

$$\left[1 - \upsilon_0' \left\{ L - \beta^2 + \frac{2k^2\beta^2}{L} \right\} \right] n^2 + \left[\upsilon_0 \left\{ L - \beta^2 + \frac{2k^2\beta^2}{L} \right\} \right] n - \frac{g\beta k^2}{L} = 0, \tag{20}$$

where

$$L = \frac{m^2\pi^2}{d^2} + k^2.$$

5. DISCUSSION

(I) **Stable case** $(\beta < 0)$. For the stable stratifications $(\beta < 0)$, if

$$\beta^2 < L + \frac{2k^2\beta^2}{L} < \beta^2 + \frac{1}{\upsilon_0'}, \tag{21}$$

all the coefficients of equation (20) are positive. Therefore, both roots of equation (20) are either real and negative or there are complex conjugates with negative real parts. The system is, thus, stable in each case.

However, for the potentially stable stratifications $(\beta < 0)$, if

$$L + \frac{2k^2\beta^2}{L} < \beta^2 , \tag{22}$$

the coefficient of n in equation (20) is negative. Therefore, there is a change of sign in the coefficients of equation (20) and hence it allows a positive root. The system is, therefore, unstable. This is in contrast to the stability of Newtonian stratified fluid, where the system is always stable for the stable stratifications.

(II) **Unstable case** $(\beta > 0)$. For the unstable stratifications $(\beta > 0)$, the constant term in equation (20) is negative. Equation (20), therefore, allows one change of sign and so has one positive root and hence the system is unstable.

We now examine the behaviours of growth rates with respect to kinematic viscosity and kinematic viscoelasticity analytically. Since for $\beta > 0$, equation (20) has one positive root, let n_0 denote the positive root, then

$$\left[1-\upsilon_0'\left\{L-\beta^2+\frac{2k^2\beta^2}{L}\right\}\right]n_0^2+\left[\upsilon_0\left\{L-\beta^2+\frac{2k^2\beta^2}{L}\right\}\right]n_0-\frac{g\beta k^2}{L}=0. \tag{23}$$

To study the behaviour of growth rates with respect to kinematic viscosity and kinematic viscoelasticity, we examine the natures of

$$\frac{dn_0}{d\upsilon_0} \quad \text{and} \quad \frac{dn_0}{d\upsilon_0'}.$$

It follows from equation (23) that

$$\frac{dn_0}{d\upsilon_0} = -\frac{L_1 n_0}{\upsilon_0 L_1 + 2n_0(1-\upsilon_0' L_1)}, \tag{24}$$

$$\frac{dn_0}{d\upsilon_0'} = \frac{L_1 n^2}{\upsilon_0 L_1 + 2n_0(1-\upsilon_0' L_1)} , \tag{25}$$

where

$$L_1 = L - \beta^2 + \frac{2k^2\beta^2}{L}.$$

It is evident from equations (24) and (25) that

$$\frac{dn_0}{d\upsilon_0} \quad \text{and} \quad \frac{dn_0}{d\upsilon_0'}$$

may be positive or negative. The growth rates, therefore, decrease as well as increase with the increase in fluid kinematic viscosity and kinematic viscoelasticity for the unstable stratifications.

CHANDRASEKHAR ([3], p. 440) has found that if n is complex, $Re(n) < 0$ if $D^2\mu$ is everywhere positive and remarked that this restriction on $D^2\mu$ is curious. Therefore, if $D^2\mu$ is negative, $Re(n)$ may be positive leading to instability. The viscosity thus has a dual role in the instability problem. The viscoelasticity may have a similar role. This may very well explain the decrease as well as increase of growth rates with the increase in fluid kinematic viscosity and kinematic viscoelasticity.

6. EFFECT OF VARIABLE HORIZONTAL MAGNETIC FIELD

Here we consider a static state in which an incompressible, infinitely electrically conducting Walters B' elastico-viscous fluid is arranged in horizontal strata in the presence of a variable horizontal magnetic field $\vec{H}(H(z), 0, 0)$. Let $\vec{h}(h_x, h_y, h_z)$ denote the perturbation in magnetic field and μ_e stands for magnetic permeability. Then the linearized hydromagnetic perturbation equations are

$$\rho\frac{\partial \vec{v}}{\partial t} = -\nabla\delta p + \vec{g}\delta\rho + \rho\left(\upsilon - \upsilon'\frac{\partial}{\partial t}\right)\nabla^2\vec{v} + \frac{\mu_e}{4\pi}[(\nabla\times\vec{h})\times\vec{H} + (\nabla\times\vec{H})\times\vec{h}]$$
$$+\left[\frac{d\mu}{dz} - \frac{\partial}{\partial t}\frac{d\mu'}{dz}\right]\left[\frac{\partial w}{\partial \bar{x}} + \frac{\partial \bar{\upsilon}}{\partial z}\right], \tag{26}$$

$$\nabla\cdot\vec{h} = 0, \tag{27}$$

$$\frac{\partial \vec{h}}{\partial t} = \nabla\times(\vec{v}\times\vec{H}) \tag{28}$$

together with equations (3) and (4). Assume that the perturbation $\vec{h}(h_x, h_y, h_z)$ in the magnetic field has also a space and time dependence of the form (8). Following the procedure as in Sections 3 and 4 and by considering stratifications in magnetic field as

$$H^2 = H_0^2 e^{\beta z} \ , \tag{29}$$

where H_0^2 is constant, we obtain

$$\left[1-\upsilon_0'\left\{L-\beta^2+\frac{2k^2\beta^2}{L}\right\}\right]n^2+\left[\upsilon_0\left\{L-\beta^2+\frac{2k^2\beta^2}{L}\right\}\right]n+\left[k_x^2V_A^2-\frac{g\beta k^2}{L}\right]=0\,, \tag{30}$$

where

$$V_A^2 = \frac{\mu_e H_0^2}{4\pi\rho_0} \text{ (square of the Alfvén velocity).}$$

7. DISCUSSION

(I) **Stable case** $(\beta < 0)$. For stable stratifications $(\beta < 0)$ as in hydrodynamic case (Section 5(I)), the system is stable if (21) is satisfied, and unstable if (22) is satisfied.

Thus, for Walters B' viscoelastic stratified fluid for stable stratifications, the system can be stable or unstable, which is in contrast to the stability of Newtonian stratified fluid in hydromagnetics where the system is always stable for the stable stratifications.

(II) **Unstable case** $(\beta > 0)$. For the unstable stratifications $(\beta > 0)$ if

$$\beta^2 < L + \frac{2k^2\beta^2}{L} < \beta^2 + \frac{1}{\upsilon_0'} \quad \text{and} \quad k_x^2 V_A^2 > \frac{g\beta k^2}{L}\,, \tag{31}$$

equation (30) does not allow of any change of sign and so has no positive root. The system is therefore stable.

But if

$$k_x^2 V_A^2 < \frac{g\beta k^2}{L}\,, \tag{32}$$

the constant term in equation (30) is negative. Equation (30), therefore, allows at least one change of sign and so has at least one positive root. The occurrence of a positive root implies that the system is unstable.

Thus, for stable stratifications, the system is found to be stable or unstable under certain conditions. However, for the unstable stratifications, the presence of the magnetic field stabilizes certain wave numbers band, whereas the system is unstable for all wave numbers in the absence of the magnetic field.

8. CONCLUSIONS

A detailed account of the stability of superposed Newtonian fluids, under varying assumptions of hydrodynamics and hydromagnetics, has been given by CHANDRASEKHAR [3]. With the growing importance of non-Newtonian fluids in chemical engineering, modern technology and industry, the investigations into such fluids are desirable. The Walters B' fluid [14] is one such important non-Newtonian (viscoelastic) fluid. WALTERS [15] reported that the mixture of polymethyl methacrylate and pyridine at 25 °C containing 30.5 g of polymer per litre with the density of 0.98 g per litre behaves in a way very similar to that of the Walters B' elastico-viscous fluid.

The Rayleigh–Taylor instability of two superposed Walters B' viscoelastic fluids has been studied. For the stable stratifications, the system is found to be stable or unstable under certain conditions. This is in contrast to the stability of two superposed Newtonian fluids where the system is stable for stable stratifications. However, the system is unstable for the unstable stratifications.

This problem has also been studied for electrically conducting Walters B' fluid in the presence of variable horizontal magnetic field. For the stable stratifications, the results are found to be the same as in the hydrodynamic case. However, for the unstable stratifications, the magnetic field has got stabilizing effects.

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