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ONE-DIMENSIONAL CONSOLIDATION OF THE POROUS MEDIUM WITH THE RHEOLOGICAL KELVIN–VOIGT SKELETON

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Abstract: In this paper, the analytical solution of porous medium consolidation with the rheological Kelvin–Voigt skeleton is presented. The rheological model is characterized by four basic physical features: elasticity, viscosity, plasticity and strength. One-dimensional problem consists in solving equations for porous column filled with liquid and being a subject of one-dimensional compression with load acting on a porous plate (allowing fluid flow), pressure gradient and weight of column itself. The results obtained may be used also for determining of the effective parameters of the Biot model. Depending on the type of equation, in the range of analytical solution, we make use of techniques based on double integral transformation of Laplace and Fourier. Within the range of boundary solutions for porous media consolidation the use is made of a finite element method.

1. INTRODUCTION

During consideration of porous medium consolidation problem, it would be necessary to obtain a solution of a boundary issue in analytical form. In such a case, there exists the possibility of analyzing any process being investigated based on mathematical analysis. In most cases, however, boundary issue is too complicated with regard to irregularity of the area investigated, irregularity of source functions, complex forms of partial differential equations and therefore it is impossible to obtain an analytical solution. In such cases, numerical methods are used for obtaining approximate solutions.

In this paper, a method for obtaining analytical solutions with the use of the Laplace transformation is presented and exemplified by the of displacements in the Biot poroelasticity model with a rheological Kelvin–Voigt skeleton. The Biot–Darcy poroelasticity model is introduced with equations for the movement of liquid and solid medium phase, for flow continuity equation and constitutive relations. This allows us to write the set of equations representing a Biot–Darcy linear consolidation theory for isothermal process that describes the displacements of the skeleton and tension in liquid. Next, the solutions obtained were compared with the results of a numerical solution.

Assume, according to STRZELECKI et al. [12], the constitutive relations for isothermal process in the form:

$$\begin{cases} \sigma_{ij} = 2N\Psi_k \varepsilon_{ij} + (A\varepsilon + Q\theta)\delta_{ij}, \\ \sigma \ = Q\varepsilon + R\theta. \end{cases} \tag{1}$$

The set of equations representing the consolidation of porous medium with a rheological Kelvin–Voigt skeleton for quasistatic processes, with the use of the Einstein index notation, might be written in the form:

$$\begin{cases} N\Psi_k \nabla^2 u_i + (M + N\Psi_k + M\Psi_k)\varepsilon,_i = -\dfrac{H}{R}\sigma,_i\,, \\ \dfrac{k}{f^2}\nabla^2\sigma = \dfrac{1}{R}\dot{\sigma} - \dfrac{H}{R}\dot{\varepsilon}, \end{cases} \tag{2}$$

where: N, M, H, R – the Biot constants; $T = \eta_s / N$ – the viscosity parameter; $\Psi_k = 1 + T\,\partial/\partial t$ – the differential operator in Kelvin–Voigt skeleton; k – the Darcy filtration coefficient; f – the skeleton porosity. η_s – the viscosity of skeleton. $\varepsilon,_i$ – the velocity of skeleton dilatation, θ – the velocity of fluid dilatation, u – the skeleton displacement, σ – the fluid stress, σ_{ij} – the skeleton stress, δ_{ij} – the Kronecker delta.

$$\begin{cases} N\nabla^2 u_i + (M + N)\,\varepsilon,_i + N\,T\,\nabla^2 \dot{u}_i + (N + M)T\dot{\varepsilon},_i = -\dfrac{H}{R}\sigma,_i\,, \\ K\nabla^2\sigma = \dfrac{1}{R}\dot{\sigma} - \dfrac{H}{R}\dot{\varepsilon}. \end{cases} \tag{3}$$

The above set of equations describes the process of consolidation caused by filtration flow of viscous Newtonian liquid passing through the pores of the Kelvin–Voigt skeleton. Solutions of this set were given by: AURIAULT et al. [1], EMMRICH [9], STRZELECKI and ŻAK [13].

2. PRELIMINARY ASSUMPTIONS OF ONE-DIMENSIONAL CONSOLIDATION MODEL WITH THE KELVIN–VOIGT RHEOLOGICAL SKELETON

The results of the consolidation process of column-shape porous medium, whose solid particles being subjected to load and hydrostatic pressure gradient, are analyzed. The examples of such a medium might be cohesive soils built of secondary minerals such as illite, montmorillonite and kaolinite. Analytical solutions of a one-dimensional problem with the classical Biot model were proposed by: AURIAULT et al. [1], JASIEWICZ [11], BAUER and STRZELECKI [2], GASZYŃSKI [10], EMMRICH [9] and DERSKI [5], [6]. As mentioned above, we assume that all kinds of loads are applied instantly at the time $t = +0$ which is represented by the Heaviside function. The subject of analysis was the consolidation caused by external load and hydrostatic pressure gradient which is schematically presented in figure 1.

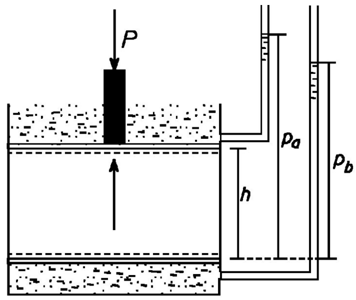

Fig. 1. Sketch of one-dimensional consolidation of the Biot skeleton

Boundary conditions:

Load condition on the upper boundary: $\sigma_{33}(h,t) = -P\eta(t)$.

Fluid stress condition on the upper boundary: $\sigma(h,t) = -P_a\eta(t)$.

Stress condition on the bottom boundary: $\sigma(0,t) = -P_b\eta(t)$.

Displacement condition on the bottom boundary: $u(0, t) = 0$.

Initial conditions: $\sigma^{(0)} - H\varepsilon^{(0)} = 0$.

Such functions as fluid stress, skeleton stress and deformation were the subject of the Laplace transformation. Image functions in the Laplace space are denoted by

$$(\tilde{\sigma}_{33}, \tilde{\sigma}, \tilde{u}, \tilde{\varepsilon}, \tilde{\theta}) = L(\sigma_{33}, \sigma, u, \varepsilon, \theta) .$$

3. ANALYTICAL SOLUTION

Taking into account the initial conditions, the set of equations for the consolidation of the porous medium with the Kelvin–Voigt skeleton in the Laplace space takes the form:

$$\begin{cases} (M + 2N + 2NTs)\dfrac{\partial^2 \tilde{u}}{\partial x^2} = -\dfrac{H}{R}\dfrac{\partial \tilde{\sigma}}{\partial x}, \\ K\dfrac{\partial^2 \tilde{\sigma}}{\partial x^2} = \dfrac{s}{R}\tilde{\sigma} - s\dfrac{H}{R}\dfrac{\partial \tilde{u}}{\partial x}. \end{cases} \tag{4}$$

In order to obtain the Laplace transformation for boundary conditions, the Heaviside function was transformed, thus boundary conditions in the Laplace space assume the form:

$$\tilde{\sigma}_{33}(h,t) = -\frac{P}{s}, \quad \tilde{\sigma}(h,t) = -\frac{P_a}{s}, \quad \tilde{\sigma}(0,t) = -\frac{P_b}{s}, \quad \tilde{u}(0,t) = 0,$$

where s is the transformation parameter.

After transformations of (4) we obtain:

$$\frac{\partial^3 \tilde{\sigma}}{\partial x^3} = P(s)\frac{\partial \tilde{\sigma}}{\partial x},$$

where:

$$P(s) = \frac{s(a+s)}{b(c+s)} = \frac{s\left[\dfrac{H^2 + R(M+2N)}{2NTR} + s\right]}{KR\left(\dfrac{M+2N}{2NT} + s\right)}. \tag{5}$$

The solution of (5) in the Laplace space according to DITKIN and PRUDNIKOW [9] is as follows:

$$\tilde{\sigma} = Ae^{x\sqrt{P(s)}} + Be^{-x\sqrt{P(s)}} + C. \tag{6}$$

After double differentiation, inserting the solution into flow equation in the set of equation (4) and double integration we obtain the function of displacements:

$$\tilde{u} = -\frac{H}{2RNT\sqrt{P(s)}(c+s)}\left(Ae^{x\sqrt{P(s)}} - Be^{-x\sqrt{P(s)}}\right) + Dx + E, \tag{7}$$

where D, E are the functions of the integration of parameter s.

Inserting functions (6) and (7) into boundary conditions, making use of $C = HD$ and constitutive relations, we obtained the set of algebraic equations used for determining the constants A, B, C, D and E. After inserting the constants into image functions (6) and (7) we arrived at final forms of equations (6) and (7). In this paper, we present the analysis for an *image function* of the displacement $\tilde{u}$:

$$\tilde{u} = \frac{H^2 P}{4RN^2T^2}\left(\frac{\cosh((x-h)\sqrt{P(s)}) + \cosh(x\sqrt{P(s)}) + \sinh(h\sqrt{P(s)}) + 2}{s\sqrt{P(s)}(c+s)(a+s)\sinh(h\sqrt{P(s)})}\right)$$

$$+ \frac{H^2(P_b - P_a)}{4RN^2T^2}\left(\frac{\cosh((x-h)\sqrt{P(s)}) + \sinh(h\sqrt{P(s)})}{s\sqrt{P(s)}(c+s)(a+s)\sinh(h\sqrt{P(s)})}\right)$$

$$-\frac{H^2(R+H)}{4R^2N^2T^2}\left(\frac{P_b\cosh((x-h)\sqrt{P(s)})+P_a\cosh(x\sqrt{P(s)})+P_b\sinh(h\sqrt{P(s)})+2P_a}{s\sqrt{P(s)}(c+s)(a+s)\sinh(h\sqrt{P(s)})}\right)$$

$$-\frac{H}{2RNT}\left(\frac{P_b\cosh((x-h)\sqrt{P(s)})+P_a\cosh(x\sqrt{P(s)})+P_b\sinh(h\sqrt{P(s)})+2P_a}{s\sqrt{P(s)}(c+s)\sinh(h\sqrt{P(s)})}\right)$$

$$+\frac{P_a-P}{s2NT(a+s)}x\,. \tag{8}$$

To find the retransformed form of displacement *image function*, the residue theory was used. Based on the Cauchy residue theorem and the Jordan lemma, the original of the rational function $\widetilde{F}(s)$ with single poles s_k is the following rational function:

$$L^{-1}[\widetilde{F}(s)]=\frac{LI(0)}{MI0}+\sum_{k=1}^{n}\frac{LI(s_k)}{s_kMI'(s_k)}e^{s_kt}\ , \tag{9}$$

where $LI(s)$ and $MI(s)$ are prime polynomials with respect to themselves, and the degree of $LI(s)$ is lower than that of $MI(s)$.

After retransformation, the function describing the displacements (u) takes the following form:

$$u=\frac{H^2P}{2hRN^2T^2}S_1+\frac{H^2P}{2hRN^2T^2}S_2-\frac{H^2(R+H)}{2hR^2N^2T^2}S_3$$

$$-\frac{H}{hRNT}S_4+\frac{x(P_a-P)}{2NTa}[1-e^{-at}]\,. \tag{10}$$

The series in equation (10) are as follows:

$$S_1=\sum_{k=1}^{n}\frac{\{(-1)^n+1\}\cos\left(\frac{n\pi}{h}x_3\right)+2}{s_kP'(s_k)(c+s_k)(a+s_k)}e^{s_kt}\ ,\qquad S_2=\sum_{k=1}^{n}\frac{(-1)^n\cos\left(\frac{n\pi}{h}x_3\right)}{s_kP'(s_k)(c+s_k)(a+s_k)}e^{s_kt}\ ,$$

$$S_3=\sum_{k=1}^{n}\frac{\{P_a(-1)^n+P_b\}\cos\left(\frac{n\pi}{h}x_3\right)+2P_a}{s_kP'(s_k)(c+s_k)(a+s_k)}e^{s_kt}\ , \tag{11}$$

$$S_4=\sum_{k=1}^{n}\frac{\{P_a(-1)^n+P_b\}\cos\left(\frac{n\pi}{h}x_3\right)+2P_a}{s_kP'(s_k)(a+s_k)}e^{s_kt}\ .$$

4. RESULTS OF SKELETON DISPLACEMENTS

Equation (10), i.e., the analytical solution, was used for calculating vertical displacements (u) at given loads: $P = 1.5 \cdot 10^5\,\mathrm{Pa}$; $P_a = 0.55 \cdot 10^5\,\mathrm{Pa}$; $P_b = 1.2 \cdot 10^5\,\mathrm{Pa}$; the sample parameters $h = 10.0\,\mathrm{m}$; $f = 0.35$ and Biot constants: $M = 5 \cdot 10^7\,\mathrm{Pa}$; $R = 1.5 \cdot 10^7\,\mathrm{Pa}$; $N = 2.5 \cdot 10^7\,\mathrm{Pa}$; $H = 2.25 \cdot 10^7\,\mathrm{Pa}$; $Q = 3.75 \cdot 10^7\,\mathrm{Pa}$. The results of calculations are presented in figure 2.

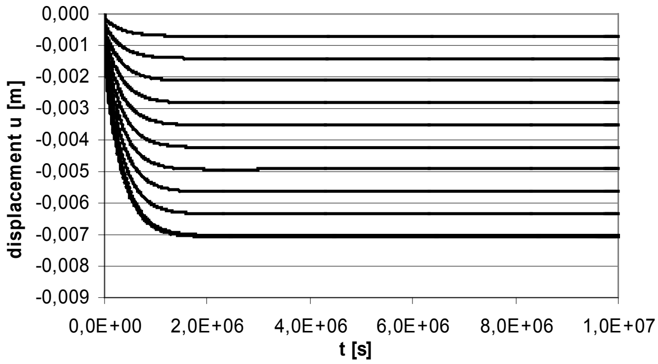

Fig. 2. Consolidation progress in the medium investigated.
The results obtained with analytical method

In order to verify the consolidation results for poroelastic Biot medium with rheological Kelvin–Voigt skeleton, they were analyzed numerically based on the one-dimensional rheological Kelvin–Voigt model, under the following boundary conditions: upper boundary $\sigma = 1.5 \cdot 10^5\,\mathrm{Pa}$, $P_a = 1.5 \cdot 10^5\,\mathrm{Pa}$, bottom boundary $P_b = 1.5 \cdot 10^5\,\mathrm{Pa}$.

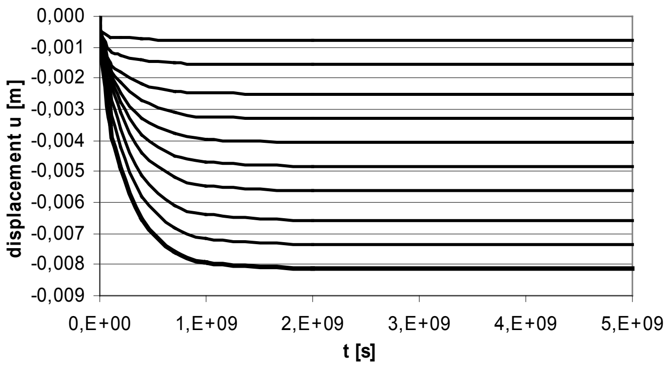

Fig. 3. Consolidation progress in the medium investigated.
The results obtained with numerical method

The sample is rigidly restricted from its bottom, which protects this bottom from vertical displacements, and is subjected to hydrostatic pressure gradient (caused by the loads P_a and P_b). The results of calculations, performed with FlexPDE program, are presented in figure 3.

5. SUMMARY AND CONCLUSIONS

The influence of pressure gradient in liquid on the creeping process of the sample obtained using analytical and numerical methods is presented in figures 2 and 3, respectively. The shapes of the curves representing the consolidation progress obtained with both methods are similar which confirms that equations obtained with analytical method are correct. Small differences in the values obtained result from numerical errors.

The plots presented differ significantly from classic Biot model, which simulates immediate settlements and might often represent significant stage of final settlement. Numerous tests performed with odeometers demonstrate that in reality we do not observe any immediate settlements of samples. The Biot–Darcy model with the Kelvin–Voigt skeleton describes the creeping process of cohesive soils.

After an appropriately long time, the creeping approaches the constant values of displacements, which means that the sample consolidation is completed. The final effect of consolidation is linearly proportional to the cross-section coordinate of the sample.

The initial conditions assumed affect significantly the consolidation computation with the numerical method. The assumption of initial settlements different from zero considerably disturbs the computing process in used by us FlexPDE program and generates significant numerical errors.

REFERENCES

[1] AURIAULT J.L., BAUER J., STRZELECKI T., *Consolidation sous elektro-osmose d'un echantillon oedometrique. Determination des constants caracteristiques*, Studia Geotechnica et Mechnica, 1981, Vol. III, No. 1.

[2] BAUER J., STRZELECKI T., *La determination des constantes de la consolidation electrohydrodynamigue*, Colloque International sur le Compactage, Ecole Nationale des Ponts et Chaussees, Laboratoire Central des Ponts et Chaussees, Paris, 1980, 22–24.

[3] BAUER J., STRZELECKI T., SZCZEŚNIAK K., *Methodes de determination des parametres pour certains modeles de milieu monophasigue et biphasique*, Comptes-rendus du 2 Coloque Franco-Polonais de Mechanigue des Sols Appliquee, INSA, Lyon, 1981, 23–28.

[4] BIOT M.A., *General Solutions of the Equations of Elasticity and Consolidation of a Porous Material*, J. Appl. Mech., 1956, 23.

[5] DERSKI W., *Some Contributions to the Theory of Flow of Fluids Through Porous Deformable Media*, Acta Mechanica, 1967,Vol. IV/1.

[6] DERSKI W., *Consolidation Displacements Produced in Elastic Porous Bodies by Fluid Sources*, Rev. Roum. de Math. Pure et Appl., 1969, Vol. XII.

[7] DERSKI W., KISIEL I., *Warunki początkowe w teorii konsolidacji*, Arch. Hydr., 1969, tom XVI.
[8] DITKIN W.A., PRUDNIKOW A.P., *Przekształcenia całkowe i rachunek operatorowy*, PWN, Warszawa 1964.
[9] EMMRICH R., *Weryfikacja doświadczalna modelu konsolidacji elektrokinetycznej*, PhD thesis, Wrocław, 1984, Politechnika Wrocławska.
[10] GASZYŃSKI J., *Identyfikacja modelu konsolidacji Biota na podstawie realizacji jednoosiowego zadania brzegowego*, Archiwum Hydrotechniki, 1984, 31.
[11] JASIEWICZ K., *Konsolidacja gruntu pod obciążeniem jego górnej powierzchni*, Archiwum Inżynierii Lądowej, 1968.
[12] STRZELECKI T., KOSTECKI S., ŻAK S., *Modelowanie przepływów przez ośrodki porowate*, Wrocław, 2007, DWE.
[13] STRZELECKI T., ZAK S., *L'Influence d'un champ electrique sur consolidation du sol*, Colloque International sur le Compactage, Paris, 1980, 22–24.