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ON THE STABILITY OF GENERALIZED HYDROMAGNETIC THERMOHALINE SHEAR FLOWS

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Streszczenie: Przedstawiono matematyczną analizę uogólnionych stabilności magnetohydrodynamicznych termohalinowych przepływów ścinających. Ich fizyczna konfiguracja jest typu poziomej warstwy, nieściśliwej, nielepkiej, przewodzącej ciepło, o zerowej rezystywności. Zachodzi w niej różnicowy przepływ $U(Z)$ w kierunku poziomym i zmienia się gęstość $\rho_0 f(Z)$ w kierunku pionowym, podczas gdy cały układ jest ograniczony dwiema poziomymi granicami o różnej, ale stałej temperaturze i stężeniu. W jednorodnym polu magnetycznym temperatura i stężenie dolnej granicy są większe niż górnej, ρ_0 jest dodatnią stałą, $U(Z)$ i $f(Z)$ są ciągłymi funkcjami współrzędnej pionowej Z, a $df/dZ < 0$ w całym obszarze przepływu. Wyprowadzono warunki dostateczne dla nadstabilności i podano ograniczenia dla dowolnie niestabilnej wartości modalnej układu w przypadku, gdy temperatura i stężenie mają odwrotne udziały w gradiencie pionowej gęstości.

Abstract: The paper presents a mathematical analysis of the stability of generalized hydromagnetic thermohaline shear flows. The physical configuration is that of a horizontal layer of an incompressible inviscid heat conducting of zero electrical resistivity in which there is differential streaming $U(Z)$ in the horizontal direction and density variation $\rho_0 f(Z)$ in the vertical direction, while the entire system is confined between two horizontal boundaries of different but uniform temperature and concentration with the temperature and concentration of the lower boundary greater than that of upper one in the presence of a uniform horizontal magnetic field, ρ_0 being a positive constant having the density and $U(Z)$ and $f(Z)$ being continuous functions of the vertical co-ordinate Z with $df/dz < 0$ everywhere in the flow domain. Sufficient conditions are derived for overstability to be valid and bounds are presented for an arbitrary unstable mode of the system for the cases where the temperature and the concentration make opposing contributions to the vertical density gradient.

Резюме: Представлен математический анализ обобщенных устойчивостей магнитогидродинамических термолинейных срезывающих течений. Их физическая конфигурация является типа горизонтального слоя, несжатая, невязкая, теплопроводная, нулевого удельного сопротивления. В ней имеется дифференциальное течение $U(Z)$ в горизонтальном направлениии и изменяется плотность $\rho_0 f(Z)$ в вертикальном направлении в то время, когда вся система ограничена двумя границами разной, но постоянной температуры и концентрации. В однороднем магнитном поле температура и концентрация нижней границы становятся выше, чем верхней, ρ_0 является положительным постоянным, $U(Z)$ и $f(Z)$ являются непрерывными функциямивертикальной координаты Z, а $df/dz < 0$ во всей зоне течения. Выведены достаточные условия для сверхустойчивости и даны ограничения для произвольно неустойчивого модального значения системы в случае, когда температура и концентрация имеют обратное участие в градиенте вертикальной плотности.

1. INTRODUCTION

The stability of parallel shear flow of an inviscid non-homogeneous fluid with stable density stratification to infinitesimal non-divergent disturbances has pervaded the scientific literature fairly recently on account of its importance in the fields of meteorology and oceanography etc. The analysis in this paper is primarily based on the fundamental works of TAYLOR [1], GOLDSTEIN [2], DRAZIN [3], MILES [4], HOWARD [5] and others on the stability of non-homogeneous shear flows, and is motivated by the concentration that in the mathematical model of the problem considered by these authors the fluid is taken to be initially non-homogeneous without assigning any reason for the cause of this initial non-homogeneity. However, the initial non-homogeneity may be due to variable temperature or concentration or some other cause. Diffusion effects, which tend to produce these changes in the density of an individual fluid particle in the course of motion, are ignored in the investigations. Therefore, it becomes important to investigate the problem by retaining the initial non-homogeneity and also taking into account the diffusion effects. GUPTA et al. [6] investigated the problem by taking into account the changes in density due to thermal effects and referred to the problem as the problem of generalized thermal shear flows.

In the present paper, sufficient conditions are derived for overstability to be valid and bounds are presented for an arbitrary unstable mode of the system by taking into account the change in density due to both thermal and concentration effects (the problem here referred as generalized thermohaline shear flows) in the presence of horizontal magnetic field.

2. MATHEMATICAL FORMULATION AND ANALYSIS

The relevant governing equations and boundary conditions of thermohaline shear flow wherein a uniform horizontal magnetic field is superimposed are given by [6]

$$(U-C)^2\left(D^2-a^2\right)\omega-(U-C)\left(D^2U\right)\omega-Q\left(D^2-a^2\right)\omega+R_3\omega$$
$$=iR_1a(U-C)\theta-iR_sa(U-C)\phi, \qquad (1)$$

$$\left\{D^2-a^2-ia(U-C)\right\}\theta=-\omega, \qquad (2)$$

$$\left\{D^2-a^2-\frac{ia}{\tau}(U-C)\right\}\phi=-\frac{\omega}{\tau}. \qquad (3)$$

where

$$C = \frac{i\sigma}{a}, \quad R_3 = \hat{R}_2 N^2, \quad \hat{R}_2 = \frac{\rho_0 d^4}{K_T^2}, \quad \text{and} \quad N^2 = \frac{-g\dfrac{df}{dz}}{\rho_0}$$

is the Brunt–Vaisala frequency. The various symbols occurring in the governing equations have their usual meanings.

The solution of equations (1)–(3) must be sought subject to the following boundary conditions:

$$\omega = \theta = \phi = 0 \quad \text{at} \quad z = 0 \quad \text{and} \quad z = 1\,. \tag{4}$$

Equations (1)–(3) together with the boundary conditions (4) present an eigenvalue problem for $C(= C_r + iC_i)$ for given values of the other parameter, and a given state of the system is stable, neutral or unstable, provided the C_i is negative, zero or positive, respectively. Further, if $C_i = 0$ implies that $C_r = 0$ for every wave number a, then the principle of exchange of stabilities (PES) is valid. Otherwise we have overstability at least when instability sets in as certain modes. It is to be noted that the inclusion of the convective effects of heat and mass transfer make the definition of stable, neutral and unstable modes distinctly clear in the sense that the existence of a stable mode is no longer implies the existence of an unstable mode etc., as is there in the classical instability problem of heterogeneous shear flows.

We prove the following theorems:

Theorem 1. If $(C, \omega, \theta, \phi)$, $C = C_r + iC_i$ is a solution of equations (1)–(4) with $R_1 > 0$, $R_s > 0$, $Q > 0$ and $Q < U_{\min}^2$ and

(i) $UD^2U > 0, \quad \forall Z \in [0,1]$,

(ii) $R_s \leq \left\{ \dfrac{|UD^2U|}{2} - \dfrac{Q}{U}\dfrac{(D^2U)}{2} - R_3 \right\}_{\min}$,

then $C_i = 0 \Rightarrow C_r \neq 0$ for some wave number a.

Proof. If possible, let $C_i = 0 \Rightarrow C_r = 0$, $\forall a$ so that $C = 0$ is allowed by the governing equations and boundary conditions.

Equations (1)–(3) then assume the form

$$U^2(D^2 - a^2)\omega - U(D^2U)\omega - Q(D^2 - a^2)\omega + R_3\omega = iR_1 a\theta U - iR_s aU\phi\,, \tag{5}$$

$$\{D^2 - a^2 - iaU\}\theta = -\omega\,, \tag{6}$$

$$\left\{D^2 - a^2 - \frac{iaU}{\tau}\right\}\phi = -\frac{\omega}{\tau}\,. \tag{7}$$

In view of condition (i) of the theorem

$$U \neq 0, \quad \forall Z \in [0,1],$$

so that equation (5) can also be written as

$$U^2(D^2 - a^2)\omega - (D^2U)\omega - Q(D^2 - a^2)\frac{\omega}{U} + R_3\frac{\omega}{U} = iR_1 a\theta - iR_s a\phi. \tag{8}$$

Multiplying equations (8), (6) and (7) by ω^*, $-iR_1 a\theta^*$ and $i\tau R_s a\phi^*$ (* indicates complex conjugation), respectively, integrating over the vertical range of Z by parts appropriately, using the boundary conditions (4) and adding the resulting equations, we arrive at

$$\int_0^1 U\left(|D\omega|^2 + a^2|\omega|^2\right)dz + \int_0^1 \omega^* DU\, D\omega\, dz + \int_0^1 (D^2U)|\omega|^2 dz$$

$$+R_1 a^2 \int_0^1 U|\theta|^2 dz + iR_s a\tau \int_0^1 \left(|D\phi|^2 + a^2|\phi|^2\right)dz - iR_1 a \int_0^1 \left(|D\theta|^2 + a^2|\theta|^2\right)$$

$$-Q\int_0^1 \frac{1}{U}\left(|D\omega|^2 + a^2|\omega|^2\right)dz + Q\int_0^1 \frac{\omega D\omega^* DU}{U^2} dz$$

$$= \int_0^1 \frac{R_3}{U}|\omega|^2 dz + 2iR_s aR_e \int_0^1 \phi\omega^* dz - 2iR_1 aR_e \int_0^1 \theta\omega^* dz, \tag{9}$$

where R_e stands for the real part.

Equating the real parts of equation (9), we have

$$\int_0^1 U\left(|D\omega|^2 + a^2|\omega|^2\right)dz + \frac{1}{2}\int_0^1 (D^2U)\left(|\omega|^2\right)dz + R_1 a^2 \int_0^1 U|\theta|^2 dz$$

$$-\frac{Q}{2}\int_0^1 \frac{|\omega|^2 D^2U}{U^2} dz + Q\int_0^1 \frac{|\omega|^2 (DU)^2}{U^3} dz - Q\int_0^1 \frac{1}{U}\left(|D\omega|^2 + a^2|\omega|^2\right)dz$$

$$= \int_0^1 \frac{R_3}{U}|\omega|^2 dz + R_s a^2 \int_0^1 U|\phi|^2 dz. \tag{10}$$

Multiplying equation (7) by its complex conjugate and integrating over the vertical range of Z, we obtain

$$\int_0^1 \frac{1}{U}\left|(D^2 - a^2)\phi\right|^2 dz + \frac{a^2}{\tau^2}\int_0^1 U|\phi|^2 dz = \frac{1}{\tau^2}\int_0^1 \frac{1}{U}|\omega|^2 dz. \tag{11}$$

Condition (i) of the theorem implies that either (a) $U > 0$, $D^2U > 0$ or (b) $U < 0$, $D^2U < 0$, $\forall z \in [0, 1]$.

If (a) holds, then equation (11) gives

$$\int_0^1 U|\phi|^2 dz < \frac{1}{a^2}\int_0^1 \frac{1}{U}|\omega|^2 dz . \tag{12}$$

Using inequality (12) in equation (10), we get

$$\int_0^1 \left(U - \frac{Q}{U}\right)\left\{|D\omega|^2 + a^2|\omega|^2\right\}dz + R_1 a^2 \int_0^1 |\theta|^2 U dz + Q\int_0^1 \frac{|\omega|^2 (DU)^2}{U^3} dz$$

$$+ \int_0^1 \frac{1}{U}\left(\frac{UD^2U}{2} - \frac{(D^2U)Q}{2U} - R_3 - R_s\right)|\omega|^2 dz < 0 . \tag{13}$$

If (b) holds, it is easily seen that inequality (13) assumes the form

$$\int_0^1 \left(|U| - \frac{Q}{|U|}\right)\left\{|D\omega|^2 + a^2|\omega|^2\right\}dz + R_1 a^2 \int_0^1 |U||\theta|^2 dz + Q\int_0^1 \frac{|\omega|^2 (DU)^2}{|U|^3}$$

$$+ \int_0^1 \frac{1}{|U|}\left(\frac{|U||D^2U|}{2} - \frac{|D^2U|Q}{2|U|} - R_3 - R_s\right)|\omega|^2 dz < 0 . \tag{14}$$

Inequalities (13)–(14) obviously cannot hold under conditions (i) and (ii) of the theorem. Hence, in the condition of the theorem $C_i = 0 \Rightarrow C_r \neq 0$ for some wave number a. This completes the proof of the theorem.

The essential content of Theorem 1, from the point of view of hydrodynamic instability, is that an arbitrary neutral mode in the problem of generalized hydromagnetic thermohaline shear flows of Veronis [7] ($R_1 > 0$, $R_s > 0$) is definitely not non-oscillatory ($C_r = 0$) in character, i.e., PES is not valid if $UD^2U > 0$, everywhere in $(0,1)$ and

(i) $Q < U^2_{\min}$,

(ii) $R_s \le \left\{\frac{UD^2U}{2} - \frac{QD^2U}{2U} - R_3\right\}_{\min}$.

Special cases. It follows from Theorem 1 that the PES is not valid for

(i) Generalized thermal shear flows ($R_s = 0 = Q$) if

$$UD^2U > 0, \quad \forall Z \in [0,1] \quad \text{and} \quad \left\{\frac{UD^2U}{2} - R_3\right\}_{\min} \geq 0.$$

(ii) Generalized hydromagnetic thermal shear flows ($R_s = 0$) if

$$UD^2U > 0, \quad \forall Z \in [0,1] \quad \text{and} \quad \left\{\frac{UD^2U}{2} - \frac{QD^2U}{2U} - R_3\right\}_{\min} \geq 0.$$

(iii) Thermal shear flows ($R_s = 0 = R_3 = Q$) if

$$UD^2U > 0, \quad \forall Z \in [0,1].$$

In fact if U is linear and $U(Z) \neq 0$, $\forall Z \in [0,1]$, then one could see from proof of the theorem that the result remains valid.

(iv) Hydromagnetic thermohaline shear flows of Veronis type ($R_3 = 0$, $R_1 > 0$, $R_s > 0$) if

$$UD^2U > 0, \quad \forall Z \in [0,1] \quad \text{and} \quad R_s \leq \left\{\frac{UD^2U}{2} - \frac{QD^2U}{2U}\right\}_{\min}.$$

(v) Thermohaline shear flows of Veronis type ($R_1 > 0$, $R_s > 0$, $R_3 = 0 = Q$) if

$$UD^2U > 0, \quad \forall Z \in [0,1] \quad \text{and} \quad R_s \leq \left\{\frac{UD^2U}{2}\right\}_{\min}.$$

(vi) Generalized thermohaline shear flows of Veronis type ($Q = 0$) if

$$UD^2U > 0, \quad \forall Z \in [0,1] \quad \text{and} \quad R_s \leq \left\{\frac{UD^2U}{2} - R_3\right\}_{\min}.$$

Theorem 2. If $((C, \omega, \theta, \phi)$, $C = C_r + iC_i$ is a solution of equations (1)–(4) with $R_1 > 0$, $R_s > 0$, $Q > 0$ and $Q < U^2_{\min}$ and

(i) $U > 0$ and $D^2U \leq 0$, $\forall Z \in [0,1]$, or (b) $U < 0$ and $D^2U \geq 0$, $\forall Z \in [0,1]$,

(ii) $R_s \leq \left\{\left\langle \pi^2 U^2 - \frac{|UD^2U|}{2}\right\rangle \left(1 - \frac{Q}{U^2}\right) - R_3\right\}_{\min}$,

then $C_i = 0 \Rightarrow C_r \neq 0$ for some wave number a.

Proof. If possible, let

$$C_i = 0 \Rightarrow C_r = 0, \quad \forall a,$$

so that $C = 0$ is allowed by the governing equations and boundary conditions. Proceeding exactly as in Theorem 1 upon considering the case (a), we have

$$\int_0^1 U\left(|D\omega|^2 + a^2|\omega|^2\right)dz + R_1a^2\int_0^1 U|\theta|^2\,dz + \frac{Q}{2}\int_0^1 \frac{|\omega|^2|D^2U|}{U^2}\,dz + Q\int_0^1 \frac{|\omega|^2|DU|^2}{U^3}\,dz$$
$$= \int_0^1 \frac{R_3}{U}|\omega|^2\,dz + R_s a^2\int_0^1 U\left(|\phi|^2\right)dz + Q\int_0^1 \frac{1}{U}\left(|D\omega|^2 + a^2|\omega|^2\right)dz + \frac{1}{2}\int_0^1 |DU|^2\,|\omega|^2\,dz. \quad (15)$$

Equation (15) together with inequality (12), yields

$$\int_0^1 U\left(|D\omega|^2 + a^2|\omega|^2\right)dz + R_1a^2\int_0^1 U|\theta|^2\,dz + \frac{Q}{2}\int_0^1 \frac{|\omega|^2|D^2U|}{U^2} + Q\int_0^1 \frac{|\omega|^2|DU|^2}{U^3}\,dz$$
$$< \int_0^1 \frac{R_3}{U}|\omega|^2\,dz + R_s\int_0^1 \frac{1}{U}|\omega|^2\,dz + \frac{1}{2}\int_0^1 |D^2U||\omega|^2\,dz + Q\int_0^1 \frac{1}{U}\left(|D\omega|^2 + a^2\,|\omega|^2\right)dz. \quad (16)$$

Now since $U > 0$, $U^2 - Q > 0$, $\forall Z \in [0,1]$, we have

$$\int_0^1 \left(\frac{U^2 - Q}{U}\right)|D\omega|^2\,dz \ge \left(\frac{U^2 - Q}{U}\right)_{\min}\int_0^1 |D\omega|^2\,dz,$$

which upon using the Poincare's inequality, namely,

$$\int_0^1 |Df_1|^2\,dz \ge \pi^2\int_0^1 |f_1|^2\,dz, \quad (17)$$

where $f_1(0) = 0 = f_1(1)$ with $f_1 = \omega$, gives

$$\int_0^1 U|D\omega|^2\,dz \ge \pi^2 U_{\min}\int_0^1 |\omega|^2\,dz. \quad (18)$$

Using inequality (18) in inequality (16), we obtain

$$\int_0^1 \frac{1}{U}\left\{U\pi^2\left\langle\frac{U^2\left(1-\frac{Q}{U^2}\right)}{U}\right\rangle_{\min} - \frac{1}{2}\left(1-\frac{Q}{U^2}\right)|UD^2U| - R_3 - R_s\right\}|\omega|^2\,dz$$

$$+R_1a^2\int_0^1 U|\theta|^2 dz + Q\int_0^1 \frac{|\omega|^2|DU|^2}{U^3}dz < 0. \tag{19}$$

Similarly in case (b), it is easily seen that inequality (19) assumes the form

$$\int_0^1 \frac{1}{|U|}\left\{|U|\pi^2\left\langle\frac{U^2\left(1-\frac{Q}{U^2}\right)}{|U|}\right\rangle_{\min} - \frac{1}{2}\left(1-\frac{Q}{U^2}\right)\left|UD^2U\right| - R_3 - R_s\right\}|\omega|^2$$

$$+R_1a^2\int_0^1 |U||\theta|^2 + Q\int_0^1 \frac{|\omega|^2|DU|^2}{|U|^3} < 0. \tag{20}$$

Inequalities (19)–(20) obviously cannot hold under the conditions of the theorem. Hence, under the conditions of the theorem $C_i = 0 \Rightarrow C_r \neq 0$ for some wave number a.

This completes the proof of the theorem.

The essential content of Theorem 2, from the point of view of hydrodynamic instability, is similar to that of Theorem 1.

Special cases. It follows from Theorem 2 that the PES is not valid for

(i) Generalized thermal shear flows ($R_s = 0 = Q$) if

$$U > 0,\quad D^2U \le 0,\quad \text{or}\quad U < 0,\quad D^2U \ge 0,\quad \forall Z \in [0,1]$$

and

$$\left\{\pi^2U^2 - \frac{1}{2}\left|UD^2U\right| - R_3\right\}_{\min} \ge 0.$$

(ii) Generalized hydromagnetic thermal shear flows ($R_s = 0$) if

$$U > 0,\quad D^2U \le 0,\quad \text{or}\quad U < 0,\quad D^2U \ge 0,\quad \forall Z \in [0,1]$$

and

$$\left\{\left\langle\pi^2U^2 - \frac{1}{2}\left|UD^2U\right|\right\rangle\left(1-\frac{Q}{U^2}\right) - R_3\right\}_{\min} \ge 0.$$

(iii) Thermal shear flows ($R_s = 0 = Q = R_3$) if

$$U > 0,\quad D^2U \le 0,\quad \text{or}\quad U < 0,\quad D^2U \ge 0,\quad \forall Z \in [0,1]$$

and

$$\left\{\pi^2 U^2 - \frac{1}{2}\left|UD^2U\right|\right\}_{\min} \geq 0 .$$

(iv) Thermohaline shear flows of Veronis type ($R_3 = 0 = Q$, $R_1 > 0$, $R_s > 0$) if

$$U > 0, \quad D^2U \leq 0, \quad \text{or} \quad U < 0, \quad D^2U \geq 0, \quad \forall Z \in [0,1]$$

and

$$R_s \leq \left\{\pi^2 U^2 - \frac{1}{2}\left|UD^2U\right|\right\}_{\min} .$$

(v) Hydromagnetic thermohaline shear flows of Veronis type ($R_3 = 0$, $R_1 > 0$, $R_s > 0$) if

$$U > 0, \quad D^2U \leq 0, \quad \text{or} \quad U < 0, \quad D^2U \geq 0, \quad \forall Z \in [0,1]$$

and

$$R_s \leq \left\{\left\langle \pi^2 U^2 - \frac{1}{2}\left|UD^2U\right|\right\rangle \left(1 - \frac{Q}{U^2}\right)\right\}_{\min} .$$

(vi) Generalized thermohaline shear flows of Veronis type ($Q = 0$) if

$$U > 0, \quad D^2U \leq 0, \quad \text{or} \quad U < 0, \quad D^2U \geq 0, \quad \forall Z \in [0,1]$$

and

$$R_s \leq \left\{\left\langle \pi^2 U^2 - \frac{1}{2}\left|UD^2U\right|\right\rangle - R_3\right\}_{\min} .$$

Theorem 3. If $((C, \omega, \theta, \phi)$, $C = C_r + iC_i$ is a solution of equations (1)–(4) with $R_1 < 0$, $R_s < 0$, $Q > 0$ and $Q < U^2_{\min}$ and

(i) $U\,D^2U \geq 0$, $\forall Z \in [0,1]$,

(ii) $\left|R_1\right| \leq \left\{\dfrac{UD^2U}{2} - \dfrac{QD^2U}{2U} - R_3\right\}_{\min}$,

then $C_i = 0 \Rightarrow C_r \neq 0$ for some wave number a.

Proof. Putting $R_1 = -\left|R_1\right|$ and $R_s = -\left|R_s\right|$ in equation (10) and using the inequality

$$\int_0^1 U|\theta|^2 dz < \frac{1}{a^2}\int_0^1 \frac{1}{U}|\omega|^2 dz. \tag{21}$$

which is derived from equation (2) in a manner similar to the derivation of inequality (12), and proceeding exactly as in Theorem 1, we get the result. This completes the proof of the theorem.

Theorem 4. If $((C,\omega,\theta,\phi), C = C_r + iC_i$ is a solution of equations (1)–(4) with $R_1 < 0, R_s < 0, Q > 0$ and $Q < U^2_{\min}$ and

(i) $U > 0,\ D^2U \le 0,\ \forall Z \in [0,1]$ or $U < 0,\ D^2U \ge 0,\ \forall Z \in [0,1]$,

(ii) $|R_1| \le \left\{\left\langle \pi^2U^2 - \frac{1}{2}|UD^2U|\right\rangle\left(1 - \frac{Q}{U^2}\right) - R_s\right\}_{\min}$,

then $C_i = 0 \Rightarrow C_r \ne 0$ for some wave number a.

Proof. Putting $R_1 = -|R_1|$ and $R_s = -|R_s|$ in equation (10), using inequalities (18) and (21) and proceeding exactly as in Theorem 2, we get the result. This completes the proof of the theorem.

The essential contents of Theorems 3 and 4, from the point of view of hydrodynamic instability, are similar to the two earlier theorems. However, presently the problem is that of generalized hydromagnetic thermohaline shear flows of Stern's [8] type ($R_1 < 0$, $R_s < 0$). Further, special cases of Theorems 3 and 4 analogous to that of the earlier theorems could be easily written down in the present case also.

Theorem 5. If $((C,\omega,\theta,\phi), C = C_r + iC_i$ is a solution of equations (1)–(4) with $R_1 > 0$, $R_s > 0$, and $Q > 0$, then $C_i < \alpha$, where α is the positive root of the cubic $\pi^2C_i^3 - \pi qC_i^2 - R_1C_i - \pi Qq = 0$, where $q = (|DU|)_{\max}$.

Proof. Since $U - C \ne 0,\ \forall Z \in [0,1]$, therefore dividing equation (1) throughout by $(U - C)$ and then proceeding as in Theorem 1, we arrive at

$$\int_0^1 (U-C)\left(|D\omega|^2 + a^2|\omega|^2\right)dz + \int_0^1 (DU)(\omega^*)(D\omega)\,dz$$
$$+ \int_0^1 (D^2U)|\omega|^2 dz + R_1a^2\int_0^1 (U-C)|\theta|^2 dz + i\tau R_s a$$

$$\cdot\int_0^1\left(|D\phi|^2+a^2|\phi|^2\right)dz-iR_1a\int_0^1\left(|D\theta|^2+a^2|\theta|^2\right)$$

$$-Q\int_0^1\frac{1}{(U-C)}\left(|D\omega|^2+a^2|\omega|^2\right)dz+Q\int_0^1\frac{\omega D\omega^* DU}{(U-C)^2}dz$$

$$=\int_0^1\frac{R_3}{(U-C)}|\omega|^2+2iR_saR_e\int_0^1\phi\omega^* dz-2iR_1aR_e\int_0^1\theta\omega^* dz. \quad (22)$$

Equating the imaginary parts of equation (22) and dividing the resulting equation throughout by $C_i(>0)$, we get

$$\int_0^1\left(|D\omega|^2+a^2|\omega|^2\right)dz+R_1a^2\int_0^1|\theta|^2dz+\frac{R_1a}{C_i}\int_1^1\left(|D\theta|^2+a^2|\theta|^2\right)$$

$$+\int_0^1\frac{R_3}{|U-C|^2}\left(|\omega|^2\right)dz+\frac{2R_sa}{C_i}R_e\int_0^1\left(\phi\omega^* dz\right)+Q\int_0^1\frac{1}{|U-C|^2}\left(|D\omega|^2+a^2|\omega|^2\right)dz$$

$$=\mathrm{I}_m\left(\frac{1}{C_i}\int_0^1(DU)(\omega^*)(D\omega)\,dz\right)+\mathrm{I}_m\left(\frac{Q}{C_i}\int_0^1\frac{UDw^* DU}{(U-C)^2}\right)$$

$$+\frac{\tau R_sa}{C_i}\int_0^1\left(|D\phi|^2+a^2|\phi|^2\right)dz+R_sa^2\int_0^1|\phi|^2dz+\frac{2R_1a}{C_i}R_e\left(\int_0^1\theta\omega^* dz\right), \quad (23)$$

where I_m stands for the imaginary part.

Using equations (2)–(3), it follows that

$$R_e\left(\int_0^1\theta\omega^* dz\right)=\int_0^0\left(|D\theta|^2+a^2|\theta|^2+aC_i|\theta|^2\right)dz \quad (24)$$

and

$$R_e\left(\int_0^1\phi\omega^* dz\right)=\tau\int_0^0\left(|D\phi|^2+a^2|\phi|^2+aC_i|\phi|^2\right)dz. \quad (25)$$

Substituting from equations (24)–(25) in equation (23) and simplifying the resulting equation, we obtain

$$\int_0^1 \left(|D\omega|^2 + a^2|\omega|^2\right)dz + \frac{\tau R_s a}{C_i}\int_0^1 \left(|D\phi|^2 + a^2|\phi|^2 + \frac{aC_i}{\tau}|\phi|^2\right)$$

$$+\int_0^1 \frac{R_3}{|U-C|^2}\left(|\omega|^2\right)dz + Q\int_0^1 \frac{1}{|U-C|^2}\left(|D\omega|^2 + a^2|\omega|^2\right)dz$$

$$= \frac{R_1 a}{C_i}\int_0^1 \left(|D\theta|^2 + a^2|\theta|^2 + aC_i|\theta|^2\right)dz + \mathrm{I}_m\left(\frac{1}{C_i}\int_0^1 \frac{(DU)(\omega^*)(D\omega)}{(U-C)^2}dz\right)$$

$$+\mathrm{I}_m\left(\frac{Q}{C_i}\int_0^1 \frac{(DU)(D\omega^*)\,\omega}{(U-C)^2}dz\right). \qquad (26)$$

Multiplying equation (2) by θ^*, integrating over the vertical range of Z by part once, using boundary conditions (4) and equating the real parts of the resulting equation, we arrive at

$$\int_0^1 \left(|D\theta|^2 + a^2|\theta|^2 + aC_i|\theta|^2\right)dz = R_e\left(\int_0^1 \omega\theta^* dz\right),$$

$$\le \left|\int_0^1 \omega\theta^* dz\right|,$$

$$\le \int_0^1 |\omega||\theta| dz,$$

$$\le \left\{\int_0^1 |\omega|^2 dz\right\}^{1/2} . \left\{\int_0^1 |\theta|^2 dz\right\}^{1/2}. \qquad (27)$$

It follows from inequality (27) that

$$aC_i\int_0^1 |\theta|^2 dz < \left\{\int_0^1 |\omega|^2 dz\right\}^{1/2} . \left\{\int_0^1 |\theta|^2 dz\right\}^{1/2}.$$

i.e.

$$\left\{\int_0^1 |\theta|^2 dz\right\}^{1/2} < \frac{1}{aC_i}\left[\int_0^1 |\omega|^2 dz\right]^{1/2}. \qquad (28)$$

Using inequality (28) in inequality (27), we obtain

$$\int_0^1 \left(|D\theta|^2 + a^2|\theta|^2 + aC_i|\theta|^2\right)dz < \frac{1}{aC_i}\int_0^1 |\omega|^2 dz,$$

which upon using inequality (17) with $f_1 = \omega$ gives

$$\int_0^1 \left(|D\theta|^2 + a^2|\theta|^2 + aC_i|\theta|^2\right)dz < \frac{1}{aC_i\pi^2}\int_0^1 |D\omega|^2 dz. \tag{29}$$

Further

$$I_m\left(\frac{1}{C_i}\int_0^1 (DU)(\omega^*)(D\omega)\,dz\right) \le \frac{1}{C_i}\int_0^1 |D\omega|\,|\omega|\,|D\omega|\,dz,$$

$$\le \frac{q}{C_i}\int_0^1 |\omega|\,|D\omega|\,dz,$$

$$\le \frac{q}{C_i}\left\{\int_0^1 |\omega|^2\,dz\right\}^{1/2}\left\{\int_0^1 |D\omega|^2\right\}^{1/2}$$

which upon using inequality (17) with $f_1 = \omega$ gives

$$I_m\left(\frac{1}{C_i}\int_0^1 (D\omega)(\omega^*)(DU)\,dz\right) \le \frac{q}{C_i\pi}\int_0^1 |D\omega|^2 dz, \tag{30}$$

where $q = \left(|DU|\right)_{\max}$ and

$$I_m\left(\frac{1}{C_i}\int_0^1 \frac{(D\omega^*)(\omega)(DU)\,dz}{(U-C)^2}\right) \le \frac{1}{C_i}\int_0^1 \frac{|D\omega|\,|\omega|\,|DU|}{(U-C)^2}$$

$$\le \frac{q}{C_i^2}\int_0^1 |\omega|\,|D\omega|\,dz$$

$$\le \frac{q}{C_i^2}\left\{\int_0^1 |\omega|^2 dz\right\}^{1/2}\left\{\int_0^1 |D\omega|^2 dz\right\}^{1/2},$$

which upon using inequality (17) with $f_1 = \omega$ gives

$$I_m\left(\frac{Q}{C_i}\int_0^1 \frac{(D\omega^*)(\omega)(DU)}{(U-C)^2}\right) \le \frac{Qq}{C_i^2\pi}\int_0^1 |D\omega|^2 dz\,, \tag{31}$$

where $q = (|DU|)_{\max}$.

Equation (26) upon using inequalities (29)–(31) gives

$$\left\{1-\frac{R}{\pi^2 C_i^2}-\frac{q}{\pi^2 C_i}-\frac{Qq}{\pi C_i^3}\right\}\int_0^1 |D\omega|^2 dz + a^2\int_0^1 |\omega|^2 dz$$

$$+\frac{\tau R_s a}{C_i}\int_0^1\left(\left(|D\phi|^2+a^2|\phi|^2\right)+\frac{aC_i}{\tau}|\phi|^2\right)dz$$

$$+\int_0^1 \frac{R_3}{|U-C|^2}|\omega|^2 dz + Q\int_0^1 \frac{1}{|U-C|^2}\left(|D\omega|^2+a^2|\omega|^2\right)dz < 0\,. \tag{32}$$

Since $a > 0$, $C_i > 0$, $\tau > 0$, $R_s > 0$ and $R_3 > 0$, therefore inequality (32) clearly implies that

$$\pi^2 C_i^3 - \pi q\, C_i^2 - R_1 C_i - \pi Q q < 0\,.$$

Hence, if α_1, α_2, α_3 are roots of this cubic, then $\alpha_1\alpha_2\alpha_3 = \pi Qq > 0 \Rightarrow$ cubic has one and only one positive root $\alpha_1 = \alpha$ (say). Thus the above cubic yields $C_i < \alpha$. This completes the proof of the theorem.

The essential content of Theorem 5, from the point of view of hydrodynamic instability, is that the growth rate of an arbitrary unstable ($C_i > 0$) mode in the problem of generalized hydromagnetic thermohaline shear flows of Veronis type ($R_1 > 0$, $R_s > 0$) is necessarily bounded with upper bound α. Further, this result is uniformly valid for the problems of hydromagnetic thermohaline shear flows, generalized hydromagnetic thermal shear flow, generalized thermohaline shear flows, etc, of Veronis type.

Theorem 6. If $((C,\omega,\theta,\phi)$, $C = C_r + iC_i$, $C_i > 0$ is a solution of equations (1)–(4) with $R_1 < 0$, $R_s < 0$ and $Q > 0$, then $C_i < \alpha$, α being the positive root of the cubic $\pi^2 C_i^3 - \pi q C_i^2 - |R_s| C_i - \pi Q q = 0$, where q is as defined in Theorem 5.

Proof. Putting $R_1 = -|R_1|$, $R_s = -|R_s|$ in equation (26), using inequalities (30), (31) and

$$\tau\int_0^1\left(|D\phi|+a^2|\phi|\right)dz + aC_i\int_0^0 |\phi|^2 < \frac{1}{aC_i\pi^2}\int_0^0 |D\omega|^2 dz\,, \tag{33}$$

which is derived from equation (3) in a manner similar to the derivation of inequality (29), we obtain the result. This completes the proof of the theorem.

The essential content of Theorem 6, from the point of view of hydrodynamic instability, is similar to that of Theorem 5. However, the problem presently is that of generalized hydromagnetic thermohaline shear flow of Stern's type.

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