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STABILITY OF SUPERPOSED VISCOUS-VISCOELASTIC (RIVLIN–ERICKSEN) FLUIDS THROUGH POROUS MEDIUM

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Streszczenie: Badano niestabilność typu Rayleigha–Taylora cieczy Newtona nakładającej się na lepkosprężystą ciecz Rivlina–Ericksena w ośrodku porowatym. Ponieważ zarówno w lepkiej cieczy Newtona, jak i w lepkiej cieczy układ jest stabilny w potencjalnie stabilnym przypadku, a niestabilny w potencjalnie niestabilnym przypadku, wniosek ten odnosi się też do badanego przez nas problemu. Rozważono oddzielnie wpływ jednorodnego poziomego pola magnetycznego i wpływ jednorodnej rotacji na niestabilność. Pole magnetyczne stabilizuje pewne pasmo liczby falowej, podczas gdy układ jest niestabilny dla wszystkich liczb falowych, gdy brak pola magnetycznego dla potencjalnie niestabilnej konfiguracji. Układ jest jednak stabilny w potencjalnie stabilnym przypadku, a niestabilny w potencjalnie niestabilnym przypadku dla bardzo lepkich cieczy poddanych jednostajnej rotacji.

Abstract: The Rayleigh–Taylor instability of a Newtonian viscous fluid overlying a Rivlin–Ericksen viscoelastic fluid through porous medium is considered. As in both Newtonian viscous–viscous fluids, the system is stable in the potentially stable case and unstable in the potentially unstable case, this holds for the present problem as well. The effects of a uniform horizontal magnetic field and a uniform rotation on the instability problem are also considered separately. The presence of magnetic field stabilizes a certain wave-number band, whereas the system is unstable for all wave-numbers in the absence of the magnetic field for the potentially unstable configuration. However, the system is stable in the potentially stable case and unstable in the potentially unstable case for highly viscous fluids in the presence of a uniform rotation.

Резюме: Исследована неустойчивость Рейлея–Тейлора ньютоновской жидкости, накладывающихся на вязкоупругую жидкость Ривлина–Эриксена в пористой среде. Из-за того, что как в вязкой ньютоновской жидкости, так и в вязкой жидкости система является устойчивой в потенциально устойчивом случае, а неустойчивой в потенциально неустойчивом, этот вывод касается также исследуемого нами вопроса. Отдельно рассмотрено влияние однородного горизонтального магнитного поля и равномерного вращательного движения на неустойчивость. Магнитное поле стабилизирует некоторую полосу волнового числа в то время, когда система является неустойчивой для всех волновых чисел, когда отсутствует магнитное поле для потенциально неустойчивой конфигурации. Система является однако устойчивой в потенциально устойчивом случае, а неустойчивой в потенциально неустойчивом случае для очень вязких жидкостей, подвергаемых равномерному вращательному движению.

1. INTRODUCTION

A detailed account of the instability of the plane interface between two Newtonian fluids, under varying assumptions of hydrodynamics and hydromagnetics, has been given by CHANDRASEKHAR [1]. BHATIA [2] has studied the influence of viscosity on the stability of the plane interface separating two incompressible superposed conducting fluids of uniform densities, when the whole system is immersed in a uniform magnetic field. He has carried out the stability analysis for two highly viscous fluids of equal kinematic viscosities and different uniform densities. RIVLIN and ERICKSEN [3] have studied the stress deformation relaxations for isotropic materials. SHARMA and KUMAR [4] have studied the hydromagnetic stability of two Rivlin–Ericksen viscoelastic superposed conducting fluids and the analysis has been carried out for two highly viscous fluids of equal kinematic viscosities and equal kinematic viscoelasticities. It is found that the stability criterion is independent of the effects of viscosity and viscoelasticity and is dependent on the orientation and magnitude of the magnetic field.

In all the above studies, the medium has been considered to be non-porous. When the fluid slowly percolates through the pores of a macroscopically homogeneous and isotropic porous medium, the gross effect is represented by Darcy's law, according to which the usual viscous term in the equations of Rivlin–Ericksen fluid motion is replaced by the resistance term

$$\left[-\frac{1}{k_1}\left(\mu+\mu'\frac{\partial}{\partial t}\right)\bar{v}\right],$$

where μ and μ' are the viscosity and viscoelasticity of the Rivlin–Ericksen fluid, k_1 is the medium permeability and $\bar{v}$ is the Darcian (filter) velocity of the fluid. LAPWOOD [5] has studied the stability of convective flow in hydromagnetics in a porous medium using Rayleigh's procedure. The Rayleigh instability of a thermal boundary layer in flow through porous medium has been considered by WOODING [6]. The thermal instability of fluids in a porous medium in the presence of suspended particles has been studied by SHARMA and SHARMA [7]. SHARMA and KUMAR [8] have studied the Rayleigh–Taylor instability of fluids in porous media in the presence of suspended particles and variable magnetic field. The instability of two viscoelastic superposed fluids with suspended particles and variable magnetic field in porous medium has been considered by KUMAR [9] who found that the stability criterion is independent of the effects of viscoelasticity, medium porosity and suspended particles but depends on the orientation and magnitude of the magnetic field.

The instability in a porous medium of a plane interface between viscous (Newtonian) and viscoelastic (Rivlin-Ericksen) fluids may find applications in geophysics, chemical technology and biomechanics and is, therefore, studied in the present paper. The effects of uniform magnetic field and uniform rotation, having relevance and importance in geophysics, are also considered. These aspects form the subject matter of the present paper.

2. PERTURBATION EQUATIONS

Consider a static state in which an incompressible Rivlin–Ericksen viscoelastic fluid is arranged in horizontal strata in porous medium, and the pressure p and the density ρ are functions of the vertical co-ordinate z only. The character of the equilibrium of this initial static state is determined, as usual, by assuming that the system is slightly disturbed and then by following its further evolution.

Let T_{ij}, τ_{ij}, e_{ij}, δ_{ij}, v_i, x_i, p, μ and μ' denote the stress tensor, shear stress tensor, rate-of-strain tensor, Kronecker delta, velocity vector, position vector, isotropic pressure, viscosity and viscoelasticity, respectively. The constitutive relations for the Rivlin–Ericksen viscoelastic fluid are

$$T_{ij} = -p\delta_{ij} + \tau_{ij} ,$$
$$\tau_{ij} = 2\left[\mu + \mu' \frac{\partial}{\partial t}\right] e_{ij} , \tag{1}$$
$$e_{ij} = \frac{1}{2}\left[\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right] .$$

Let $\vec{v}(u, v, w)$, ρ, p, ε and k_1 denote the velocity of fluid, density, pressure, medium porosity and medium permeability, respectively. Then the equations of motion and continuity for Rivlin–Ericksen incompressible viscoelastic fluid in a porous medium are

$$\frac{\rho}{\varepsilon}\left[\frac{\partial \vec{v}}{\partial t} + \frac{1}{\varepsilon}(\vec{v}\cdot\nabla)\vec{v}\right] = [-\nabla p + \rho\vec{g}] - \frac{\rho}{k_1}\left[\upsilon + \upsilon' \frac{\partial}{\partial t}\right]\vec{v} , \tag{2}$$

$$\nabla\cdot\vec{v} = 0 , \tag{3}$$

where $\vec{g}(0,0,-g)$ is the acceleration due to gravity, $\upsilon\left(=\frac{\mu}{\rho}\right)$ is kinematic viscosity of the fluid and $\upsilon'\left(=\frac{\mu'}{\rho}\right)$ is kinematic viscoelasticity of the fluid.

Since the density of a fluid particle remains unchanged as we follow it with its motion, we have

$$\varepsilon\frac{\partial \rho}{\partial t} + (\vec{v}\cdot\nabla)\,\rho = 0 . \tag{4}$$

Let $\vec{v}(u, v, w)$, $\delta\rho$ and δp denote respectively the perturbations in fluid velocity (0, 0, 0), fluid density ρ and fluid pressure p. Then the linearized perturbed forms of equations (2)–(4) become

$$\frac{\rho}{\varepsilon}\frac{\partial \vec{v}}{\partial t} = -\nabla \delta p + \vec{g}\delta\rho - \frac{\rho}{k_1}\left(\upsilon + \upsilon' \frac{\partial}{\partial t}\right)\vec{v}, \tag{5}$$

$$\nabla \cdot \vec{v} = 0 , \tag{6}$$

$$\varepsilon \frac{\partial}{\partial t}(\delta\rho) = -wD\rho , \tag{7}$$

where $D = \dfrac{d}{dz}$.

Analyzing the disturbances into normal modes, we seek solutions whose dependence on x, y and t is given by

$$\exp(ik_x x + ik_y y + nt), \tag{8}$$

where k_x, k_y are horizontal wave numbers, $k^2 = k_x^2 + k_y^2$, and n is the rate at which the system departs from the equilibrium. For perturbations of the form (8), equations (5) – (7) give

$$\frac{1}{\varepsilon}\rho n u = -ik_x \delta p - \frac{\rho}{k_1}(\upsilon + \upsilon' n)u, \tag{9}$$

$$\frac{1}{\varepsilon}\rho n v = -ik_y \delta p - \frac{\rho}{k_1}(\upsilon + \upsilon' n)v, \tag{10}$$

$$\frac{1}{\varepsilon}\rho n w = -D\delta p - g\delta\rho - \frac{\rho}{k_1}(\upsilon + \upsilon' n)w, \tag{11}$$

$$ik_x u + ik_y v + Dw = 0 , \tag{12}$$

$$\varepsilon n \delta\rho = -wD\rho . \tag{13}$$

Eliminating δp between equations (9)–(11) with the help of equations (12) and (13), we obtain

$$\frac{n}{\varepsilon}\left[D(\rho Dw) - k^2 \rho w\right] + \frac{1}{k_1}\left[D\{\rho(\upsilon + \upsilon' n)Dw\} - k^2 \rho(\upsilon + \upsilon' n)w\right] + \frac{gk^2}{\varepsilon n}(D\rho)w = 0 . \tag{14}$$

3. TWO UNIFORM RIVLIN–ERICKSEN VISCOELASTIC FLUIDS SEPARATED BY A HORIZONTAL BOUNDARY

Consider the case of two uniform fluids of densities, kinematic viscosities ρ_2, υ_2 (upper, Newtonian fluid) and ρ_1, υ_1 (lower, Rivlin–Ericksen viscoelastic fluid) separated by a horizontal boundary at $z = 0$. Then in each region of constant ρ, constant υ and constant υ', equation (14) reduces to

$$\left(D^2 - k^2\right)w = 0 . \tag{15}$$

The general solution of equation (15) is

$$w = Ae^{+kz} + Be^{-kz} , \tag{16}$$

where A and B are arbitrary constants. The boundary conditions to be satisfied in the present problem are as follows:

(i) The velocity w should vanish when $z \to +\infty$ (for the upper fluid) and $z \to -\infty$ (for the lower fluid).
(ii) $w(z)$ is continuous at $z = 0$.
(iii) The jump condition at the interface $z = 0$ between the fluids. This jump condition is obtained by integrating equation (14) over an infinitesimal element of z including 0, and is

$$\frac{n}{\varepsilon}\left[\rho_2 Dw_2 - \rho_1 Dw_1\right]_{z=0} + \frac{1}{k_1}\left[\mu_2 Dw_2 - \left(\mu_1 + \mu_1' n\right)Dw_1\right]_{z=0} = -\frac{gk^2}{\varepsilon n}\left[\rho_2 - \rho_1\right]w_0 . \tag{17}$$

Such a configuration that upper fluid is Newtonian and lower fluid is Rivlin–Ericksen viscoelastic should be remembered. Here w_0 is the common value of w at $z = 0$.

Applying the boundary conditions (i) and (ii), we can write

$$w_1 = Ae^{+kz} \qquad (z < 0) , \tag{18}$$

$$w_2 = Ae^{-kz} \qquad (z > 0) , \tag{19}$$

where the same constant A has been chosen to ensure the continuity of w at $z = 0$. Applying the condition (17) to the solutions (18) and (19), we obtain

$$\left[1 + \frac{\varepsilon}{k_1}\upsilon_1'\alpha_1\right]n^2 + \frac{\varepsilon}{k_1}\left[\upsilon_2\alpha_2 + \upsilon_1\alpha_1\right]n - gk\left[\alpha_2 - \alpha_1\right] = 0, \tag{20}$$

where

$$\alpha_{1,2} = \frac{\rho_{1,2}}{\rho_1 + \rho_2}, \quad \upsilon_1 = \frac{\mu_1}{\rho_1}, \quad \upsilon_2 = \frac{\mu_2}{\rho_2}, \quad \upsilon_1' = \frac{\mu_1'}{\rho_1} .$$

DISCUSSION

(a) Stable case $(\alpha_2 < \alpha_1)$. For the potentially stable case $(\alpha_2 < \alpha_1)$, equation (20) does not admit of any change of sign and so has no positive root. The system is therefore stable.

(b) Unstable case $(\alpha_2 > \alpha_1)$. Now for the potentially unstable case $(\alpha_2 > \alpha_1)$, the constant term in equation (20) is negative. Equation (20), therefore, allows one change of sign and so has one positive root and hence the system is unstable.

Therefore, the system is unstable for unstable configuration.

4. EFFECT OF A HORIZONTAL MAGNETIC FIELD

Consider the motion of incompressible, infinitely conducting Newtonian and Rivlin–Ericksen viscoelastic fluids in porous medium in the presence of a uniform horizontal magnetic field $\vec{H}(H,0,0)$. Let $\vec{h}(h_x,h_y,h_z)$ denote the perturbation in the magnetic field, then the linearized perturbation equations are

$$\frac{\rho}{\varepsilon}\frac{\partial \vec{v}}{\partial t} = -\nabla\delta p + \vec{g}\delta\rho + \frac{\mu_e}{4\pi}\left(\nabla\times\vec{h}\right)\times\vec{H} - \frac{\rho}{k_1}\left(\upsilon + \upsilon'\frac{\partial}{\partial t}\right)\vec{v}, \tag{21}$$

$$\nabla\cdot\vec{h} = 0, \tag{22}$$

$$\varepsilon\frac{\partial \vec{h}}{\partial t} = \nabla\times\left(\vec{v}\times\vec{H}\right), \tag{23}$$

together with equations (6) and (7). Assume that the perturbation $\vec{h}(h_x,h_y,h_z)$ in the magnetic field has also a space and time dependence of the form (8). Here μ_e stands for the magnetic permeability. Following the procedure as in Section 3, we obtain

$$\left[1 + \frac{\varepsilon}{k_1}\upsilon_1'\alpha_1\right]n^2 + \frac{\varepsilon}{k_1}\left[\upsilon_2\alpha_2 + \upsilon_1\alpha_1\right]n + \left[2k_x^2V_A^2 - gk(\alpha_2 - \alpha_1)\right] = 0, \tag{24}$$

where

$$V_A = \sqrt{\frac{\mu_e H^2}{4\pi(\rho_1 + \rho_2)}}$$

is the Alfvén velocity.

DISCUSSION

(a) **Stable case** $(\alpha_2 < \alpha_1)$. For the potentially stable case $(\alpha_2 < \alpha_1)$, equation (24) does not allow any positive root as there is no change of sign. The system is therefore stable.

(b) **Unstable case** $(\alpha_2 > \alpha_1)$. For the potentially unstable case $(\alpha_2 > \alpha_1)$, if

$$2k_x^2 V_A^2 > gk(\alpha_2 - \alpha_1),$$

i.e. if

$$\rho_2 - \rho_1 < \frac{\mu_e H^2 k_x^2}{2\pi g k}\,, \tag{25}$$

equation (24) does not admit of any change of sign and so has no positive root. The system is therefore stable.

But if

$$2k_x^2 V_A^2 < gk(\alpha_2 - \alpha_1), \tag{26}$$

the constant term in equation (24) is negative. Equation (24), therefore, allows at least one change of sign and so has at least one positive root. The occurrence of a positive root implies that the system is unstable.

Thus, for the potentially unstable configuration, the presence of magnetic field stabilizes certain wave-numbers band, whereas system was unstable for all wave numbers in the absence of magnetic field.

5. EFFECT OF UNIFORM ROTATION

Consider the motion of an incompressible Rivlin–Ericksen viscoelastic fluid in porous medium in the presence of a uniform rotation $\vec{\Omega}(0,0,\Omega)$. Then the linearized perturbation equations are

$$\frac{\rho}{\varepsilon}\frac{\partial \vec{v}}{\partial t} = -\nabla\delta p + \vec{g}\delta\rho + \frac{2\rho}{\varepsilon}\left(\vec{v}\times\vec{\Omega}\right) - \frac{\rho}{k_1}\left(\upsilon + \upsilon'\frac{\partial}{\partial t}\right)\vec{v}, \tag{27}$$

together with equations (6) and (7).

Following the same procedure as in Section 3 (and CHANDRASEKHAR [1], p. 453), we obtain

$$1 + \frac{4\Omega^2}{\left[n + \frac{\varepsilon}{k_1}(\upsilon + \upsilon' n)\right]^2} + \frac{gk^2(\alpha_1 - \alpha_2)}{n\left[n + \frac{\varepsilon}{k_1}(\upsilon + \upsilon' n)\right]\kappa} = 0, \tag{28}$$

where

$$\kappa = \frac{k}{1+\dfrac{2\Omega^2}{\left[n+\dfrac{\varepsilon}{k_1}(\upsilon+\upsilon' n)\right]^2}}, \tag{29}$$

for highly viscous fluid and viscoelastic fluids. Here we assumed the kinematic viscosities and kinematic viscoelasticities of both fluids to be equal, i.e., $\upsilon_1 = \upsilon_2 = \upsilon$ [CHANDRASEKHAR [1], p. 443), $\upsilon_1' = \upsilon'$, as these simplifying assumptions do not obscure any of the essential features of the problem. Equation (28), after substituting the value of κ from (29) and simplification, yields

$$\left[\left(1+\frac{\varepsilon\upsilon'}{k_1}\right)^3\right]n^4 + \left[\frac{3\varepsilon\upsilon}{k_1}\left(1+\frac{\varepsilon\upsilon'}{k_1}\right)^2\right]n^3 + \left[1+\frac{\varepsilon\upsilon'}{k_1}\right]$$
$$\cdot\left[\left(\frac{3\varepsilon^2\upsilon'^2}{k_1^2}+4\Omega^2\right)+\left(1+\frac{\varepsilon\upsilon'}{k_1}\right)gk(\alpha_1-\alpha_2)\right]n^2$$
$$+\left[\frac{\varepsilon^3\upsilon^3}{k_1^3}+4\Omega^2\frac{\varepsilon\upsilon}{k_1}+2\frac{\varepsilon\upsilon}{k_1}\left(1+\frac{\varepsilon\upsilon'}{k_1}\right)gk(\alpha_1-\alpha_2)\right]n$$
$$+\left[\left(\frac{\varepsilon^2\upsilon^2}{k_1^2}+2\Omega^2\right)gk(\alpha_1-\alpha_2)\right]=0. \tag{30}$$

DISCUSSION

(a) Stable case $(\alpha_2<\alpha_1)$. For the potentially stable arrangement $(\alpha_2<\alpha_1)$, all the coefficients of equation (30) are positive. So, all the roots of equation (30) are either real and negative, or there are complex roots (which occur in pairs) with negative real parts and the rest negative real roots. The system is therefore stable in each case.

(b) Unstable case $(\alpha_2>\alpha_1)$. For the potentially unstable arrangement $(\alpha_2>\alpha_1)$, the constant term in equation (30) is negative. Equation (30), therefore, allows at least one change of sign and so has at least one positive root. The system is therefore unstable for potentially unstable case.

Thus the effect of uniform rotation on the motion of an incompressible viscous fluid overlying Rivlin–Ericksen viscoelastic fluid through a porous medium makes the system stable for potentially stable cases and unstable for potentially unstable cases.

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