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STABILITY OF STRATIFIED WALTERS' (MODEL B') VISCOELASTIC FLUID IN STRATIFIED POROUS MEDIUM

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Streszczenie: Przedstawiono liniową stabilność uwarstwionej cieczy Waltersa (model B') w uwarstwionym ośrodku porowatym. Rozważono przypadki zmian wykładniczych gęstości, lepkości, lepkosprężystości, porowatości i przepuszczalności środowiska. Stwierdzono, że w przypadku potencjalnie stabilnego uwarstwienia układ jest stabilny lub niestabilny w zależności od kinematycznej lepkosprężystości, która może osiągać wartość mniejszą lub większą od wartości otrzymanej przez podzielenie przepuszczalności środowiska przez jego porowatość. Stanowi to przeciwieństwo stabilności uwarstwionej cieczy Newtona. Jednakże układ jest niestabilny w przypadku zakłóceń wszystkich liczb falowych dla potencjalnie niestabilnego uwarstwienia. Jeżeli wprowadzi się pewne ograniczenia, to szybkość wzrostu zwiększa się lub zmniejsza wraz ze zwiększającymi się wartościami parametrów uwarstwienia. Oddzielnie omówiono wpływ zmiennego poziomego pola magnetycznego i jednorodnej rotacji. Szybkość wzrostu stabilności w zależności od prędkości Alfvéna (w przypadku pola magnetycznego) oraz w zależności od prędkości kątowej (w przypadku rotacji) zbadano analitycznie i stwierdzono, że w pewnych warunkach zarówno pole magnetyczne, jak i rotacja mają sprzężony wpływ na stabilność.

Abstract: The linear stability of stratified Walters' (model B') fluid in stratified porous medium is presented. The case of exponentially varying density, viscosity, viscoelasticity, medium porosity and medium permeability is considered. It is found that for the potentially stable stratifications the system is stable or unstable, depending on the kinematic viscoelasticity which can be smaller or greater than the medium permeability divided by medium porosity. This is in contrast to the stability of stratified Newtonian fluid. However, the system is found to be unstable for disturbances of all wave numbers for potentially unstable stratifications. If some restrictions are imposed, then the growth rates are found to increase or decrease with increasing values of stratification parameters. The effects of variable horizontal magnetic field and uniform vertical rotation have also been discussed separately. The behaviour of growth rate with respect to the Alfvén velocity (in the case of magnetic field) and angular velocity (in the case of rotation) are examined analytically and it is found that under certain conditions both magnetic field and rotation have a dual effect on this stability problem.

Резюме: Представлена линейная устойчивость слоистой жидкости Вальтерса (модель Б') в слоистой пористой среде. Рассмотрен случай экспоненциально изменяющихся густоты, вязкости, вязкоупругости, а также пористости и проницаемости среды. Было установлено, что в случае

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потенциально устойчивой слоистости система является устойчивой или неустойчивой в зависимости от кинематической вязкоупругости, которая может достигать меньшего или высшего значения, чем значение, полученное посредством деления проницаемости среды на ее пористость. Это составляет противоположность устойчивости слоистой жидкости Ньютона. Однако система является неустойчивой в случае помех всех волновых чисел для потенциально неустойчивой слоистости. Если ввести некоторые ограничения, скорость повышения устойчивости растет или понижается вместе с повышающимися значениями параметров слоистости. Отдельно обсуждено влияние изменяющегося горизонтального магнитного поля и равномерного вертикального вращения. Быстрота повышения устойчивости по отношению к скорости Алфвена (в случае магнитного поля) и угловой скорости (в случае вращательного движения) были исследованы аналитически и было установлено, что в некоторых условиях как магнитное поле, так и вращательное движение имеют двойное влияние на устойчивость.

1. INTRODUCTION

The stability derived from the character of the equilibrium of an incompressible heavy fluid of variable density (i.e. of a heterogeneous fluid) is termed as the Rayleigh–Taylor instability. The Rayleigh–Taylor instability of a Newtonian fluid has been studied by several authors accepting varying assumptions of hydrodynamics and hydromagnetics, and CHANDRASEKHAR [1] in his celebrated monograph has given a detailed account of these investigations. The Rayleigh–Taylor instability problems arise in oceanography, limnology and engineering. The problem of the Rayleigh-Taylor instability of fluids in a porous medium is of great importance in geophysics, soil sciences, groundwater hydrology and astrophysics.

With the growing importance of non-Newtonian fluids in modern technology and industry, the investigations of such fluids are desirable. There are many viscoelastic fluids that cannot be characterized either by Maxwell's constitutive relations or by Oldroyd's constitutive relations. One of such viscoelastic fluids is Walters' (model B′) fluid. In recent years, the investigation of flow of fluids through porous media has become an important topic due to the recovery of crude oil from the pores of reservoir rocks. The flow through porous media is of a considerable interest for petroleum engineers and for geophysical fluid dynamicists [2]. A great number of applications of such a flow in geophysics may be found in a book by PHILLIPS [3]. When the fluid slowly percolates through the pores of rock, the gross effect is represented by Darcy's law. As a result of this macroscopic law, the usual viscous and viscoelastic terms in the equation of Walters' (model B′) fluid motion are replaced by the resistance term

$$\left[-\frac{1}{k_1}\left(\mu-\mu'\frac{\partial}{\partial t}\right)\mathbf{u}\right],$$

where μ and μ' are the viscosity and viscoelasticity of the Walters' (model B′) fluid, k_1 is the medium permeability and $\mathbf{u}$ is the Darcian (filter) velocity of the fluid.

SHARMA et al. [4], [5] have studied the thermosolutal convection in Walters' (model B′) fluid in porous medium in the presence of rotation and magnetic field, respectively.

Generally, it is accepted that comets consist of a dusty *snowball*, being a mixture of frozen gases which, in the process of their journey, changes from solid to gas and *vice versa*. The physical properties of comets, meteorites and interplanetary dust strongly suggest the importance of porosity in astrophysical context (MCDONNELL [6]). WOODING [7] has considered the Rayleigh instability of a thermal boundary layer in flow through a porous medium. In stellar interiors and atmospheres, the magnetic field may be (and quite often is) variable (and non-uniform) and may altogether alter the nature of the instability. SHARMA and SUNIL [8] have studied the Rayleigh–Taylor instability of a partially ionized plasma in a porous medium in the presence of a variable magnetic field. SHARMA et al. [9] have studied the thermal convection in Walters' viscoelastic fluid B′ that with suspended particles permeates through porous medium. There is growing importance of non-Newtonian fluids in chemical technology, industry and the dynamics of geophysical fluids.

To the best of our knowledge, the separate effects of magnetic field and rotation on stratified Walters' (model B′) fluid in stratified porous medium have not been investigated yet. So keeping in mind the importance of non-Newtonian fluids in modern technology and their various applications mentioned above, the present paper is devoted to the consideration of the stability of stratified Walters' (model B′) fluid in stratified porous medium. The effects of variable magnetic field and uniform rotation, bearing relevancy in geophysics, are also considered separately.

2. FORMULATION OF THE PROBLEM AND PERTURBATION EQUATIONS

Let us consider a static state in which an incompressible Walters' (model B′) fluid layer of variable density and viscosity is arranged in horizontal strata in porous medium of variable permeability and porosity, and the pressure p and the density ρ are functions of the vertical coordinate z only. The character of the equilibrium of this initial state is determined by supposing that the system is slightly disturbed and then by following its further evolution.

Let p, ρ, μ, μ' and $\mathbf{u}(u, v, w)$ denote, respectively, the pressure, density, viscosity, viscoelasticity and filter velocity of pure Walters' (model B′) fluid. Here g, k_1 and ε stand for acceleration due to gravity, medium permeability and medium porosity, $\mathbf{x} = (x, y, z)$ and $\boldsymbol{\lambda} = (0, 0, 1)$. Then the equations of motion and continuity for Walters' (model B′) fluid are

$$\frac{\rho}{\varepsilon}\left[\frac{\partial \mathbf{u}}{\partial t}+\frac{1}{\varepsilon}(\mathbf{u}.\nabla)\mathbf{u}\right]=-\nabla p-\rho g \lambda-\frac{1}{k_1}\left(\mu-\mu'\frac{\partial}{\partial t}\right)\mathbf{u}, \tag{1}$$

$$\nabla.\mathbf{u}=0. \tag{2}$$

Since the density of a fluid particle remains unchanged as we follow it with its motion, we have

$$\varepsilon\frac{\partial \rho}{\partial t}+(\mathbf{u}.\nabla)\rho=0. \tag{3}$$

Let us consider a small perturbation of the steady state solution, and let δp, $\delta\rho$ and $\mathbf{u}(u,v,w)$ denote, respectively, the perturbations of the pressure p, the density ρ and the fluid velocity (0, 0, 0). Then the linearized perturbation equations governing the motion of Walters' (model B′) fluid layer through porous medium are

$$\frac{\rho}{\varepsilon}\frac{\partial \mathbf{u}}{\partial t}=-\nabla\delta p-g\delta\rho\,\lambda-\frac{1}{k_1}\left(\mu-\mu'\frac{\partial}{\partial t}\right)\mathbf{u}, \tag{4}$$

$$\nabla.\mathbf{u}=0, \tag{5}$$

$$\varepsilon\frac{\partial}{\partial t}\delta\rho=-w(D\rho). \tag{6}$$

3. DISPERSION RELATION

Analyzing the perturbations into normal modes, we assume that the perturbation quantities have an x-, y- and t-dependence of the form

$$\exp(ik_x x+ik_y y+nt), \tag{7}$$

where k_x, k_y are the wave numbers along the x- and y-directions, respectively, $k=\sqrt{(k_x^2+k_y^2)}$ is the resultant wave number of disturbance and n is the growth rate which is, in general, a complex constant.

For perturbations of the form (7), equations (4)–(6) give

$$\left[\frac{n}{\varepsilon}+\frac{1}{k_1}(\nu-\nu' n)\right]\rho u=-ik_x\delta p, \tag{8}$$

$$\left[\frac{n}{\varepsilon}+\frac{1}{k_1}(\nu-\nu' n)\right]\rho v=-ik_y\delta p, \tag{9}$$

$$\left[\frac{n}{\varepsilon}+\frac{1}{k_1}(\nu-\nu' n)\right]\rho w=-D\delta p-g\delta\rho\,, \tag{10}$$

$$ik_x u+ik_y \upsilon+Dw=0\,, \tag{11}$$

$$\varepsilon n\delta\rho=-w(D\rho), \tag{12}$$

where

$$\nu=\frac{\mu}{\rho},\quad \nu'=\frac{\mu'}{\rho}\quad\text{and}\quad D=\frac{d}{dz}.$$

Multiplying (8) by $-ik_x$, (9) by $-ik_y$, adding and using (11), we obtain

$$\left[\frac{n}{\varepsilon}+\frac{1}{k_1}(\nu-\nu' n)\right]\rho Dw=-k^2\delta p\,. \tag{13}$$

Eliminating δp from (10) and (13) and using (12), we arrive at

$$\frac{n}{\varepsilon}\left[D(\rho Dw)-k^2\rho w\right]+\frac{1}{k_1}\left[D\{(\mu-\mu' n)Dw\}-k^2(\mu-\mu' n)w\right]+\frac{gk^2}{\varepsilon n}(D\rho)w=0\,. \tag{14}$$

4. THE CASE OF EXPONENTIALLY VARYING STRATIFICATIONS

In order to obtain the solution of the stability problem of a layer of Walters' (model B′) fluid, we suppose that the density ρ, viscosity μ, viscoelasticity μ', medium porosity and medium permeability vary exponentially along the vertical direction, i.e.

$$\rho=\rho_0 e^{\beta z},\quad \mu=\mu_0 e^{\beta z},\quad \mu'=\mu_0' e^{\beta z},\quad \varepsilon=\varepsilon_0 e^{\beta z}\quad\text{and}\quad k_1=k_{10}e^{\beta z}, \tag{15}$$

where ρ_0, μ_0, μ_0', ε_0, k_1 and β are constants, thus

$$\nu\left(=\frac{\mu}{\rho}=\frac{\mu_0}{\rho_0}\right)\quad\text{and}\quad \nu'\left(=\frac{\mu'}{\rho}=\frac{\mu_0'}{\rho_0}\right),$$

the coefficients of kinematic viscosity and kinematic viscoelasticity, respectively, are constant everywhere.

Using the stratification of the form (15), equation (14) transforms to

$$D^2w + \beta Dw - \frac{k^2\left[n + \frac{\varepsilon_0}{k_{10}}(\nu_0 - \nu_0' n) - \frac{g\beta}{n}\right]}{\left[n + \frac{\varepsilon_0}{k_{10}}(\nu_0 - \nu_0' n)\right]} w = 0 . \quad (16)$$

We assume that the system is confined between two rigid planes $z = 0$ and $z = d$, then the vanishing of w at $z = 0$ is satisfied by the choice

$$w = A(e^{m_1 z} - e^{m_2 z}), \quad (17)$$

where

$$m_{1,2} = -\frac{\beta}{2} \pm \frac{1}{2}\left[\beta^2 + \frac{4k^2\left\{n + \frac{\varepsilon_0}{k_{10}}(\nu_0 - \nu_0' n) - \frac{g\beta}{n}\right\}}{\left\{n + \frac{\varepsilon_0}{k_{10}}(\nu_0 - \nu_0' n)\right\}}\right]^{1/2}, \quad (18)$$

while the vanishing of w at $z = d$ requires

$$e^{(m_1 - m_2)d} = 1, \quad (19)$$

which implies that

$$(m_1 - m_2)\, d = 2is\pi, \quad (20)$$

where s is an integer.

Inserting the values of m_1 and m_2 from equation (18) into equation (20) and simplifying the latter, we obtain

$$\left[n^2\left(1 - \frac{\nu_0'\varepsilon_0}{k_{10}}\right) + n\left(\frac{\nu_0\varepsilon_0}{k_{10}}\right)\right]\left(4k^2d^2 + \beta^2d^2 + 4s^2\pi^2\right) - 4k^2d^2g\beta = 0 . \quad (21)$$

Equation (21) is the dispersion relation governing the effects of viscoelasticity, porosity, medium permeability on stability of stratified (exponentially varying density, viscosity and kinematic viscosity) viscoelastic (Walters' B′) fluids in stratified (exponentially varying medium porosity and medium permeability) porous medium.

(a) **Stable stratifications** $(\beta < 0)$. For the stable stratification $(\beta < 0)$, equation (21) does not admit any positive root of n, if

$$\frac{\varepsilon_0}{k_{10}}\nu_0' < 1, \quad (22)$$

i.e. if

$$\nu_0' < \frac{k_{10}}{\varepsilon_0}. \tag{23}$$

So, the system is stable for disturbances of all wave numbers if (23) holds.
And if

$$\nu_0' > \frac{k_{10}}{\varepsilon_0}, \tag{24}$$

then equation (21) has one change in sign implying the existence of one positive root of n, and so the system is unstable for disturbances of all wave numbers.

Thus for stable stratifications $(\beta < 0)$, the system is stable or unstable, depending on

$$\nu_0' < \quad \text{or} \quad > \frac{k_{10}}{\varepsilon_0}. \tag{25}$$

(b) **Unstable stratifications** $(\beta > 0)$. For the unstable stratifications $(\beta > 0)$, equation (21) has at least one change in sign implying the existence of at least one positive root of n, and so the system is unstable for disturbances of all wave numbers.

Thus if $\beta > 0$, equation (21) has one positive root. Let n_0 denote the positive root of (21). Then

$$\left[n_0^2\left(1 - \frac{\nu_0'\varepsilon_0}{k_{10}}\right) + n_0\left(\frac{\nu_0\varepsilon_0}{k_{10}}\right)\right]\left(4k^2d^2 + \beta^2d^2 + 4s^2\pi^2\right) - 4k^2d^2g\beta = 0. \tag{26}$$

To study the behaviour of the growth rate of unstable modes with respect to kinematic viscosity, kinematic viscoelasticity and medium permeability, we examine analytically the natures of $dn_0/d\nu_0$, $dn_0/d\nu_0'$ and dn_0/dk_{10}. Equation (26) yields

$$\frac{dn_0}{d\nu_0} = -\frac{\varepsilon_0 n_0}{\left[(k_{10} - \varepsilon_0\nu_0')2n_0 + \varepsilon_0\nu_0\right]}, \tag{27}$$

$$\frac{dn_0}{d\nu_0'} = \frac{\varepsilon_0 n_0^2}{\left[(k_{10} - \varepsilon_0\nu_0')2n_0 + \varepsilon_0\nu_0\right]}, \tag{28}$$

$$\frac{dn_0}{dk_{10}} = \frac{\varepsilon_0 n_0(\nu_0 - \nu_0' n_0)}{k_{10}\left[(k_{10} - \varepsilon_0\nu_0')2n_0 + \varepsilon_0\nu_0\right]}. \tag{29}$$

It is evident from (27) and (28) that if

$$\nu'_0 < \frac{k_{10}}{\varepsilon_0}, \tag{30}$$

then $dn_0 / d\nu_0$ and $dn_0 / d\nu'_0$ are always negative and positive, respectively. The growth rates, therefore, decrease with the increase in kinematic viscosity and increase with the increase in kinematic viscoelasticity. If

$$\nu'_0 > \frac{k_{10}}{\varepsilon_0}, \tag{31}$$

then $dn_0 / d\nu_0$ and $dn_0 / d\nu'_0$ are negative or positive and positive or negative, respectively, if

$$\left|(k_{10} - \varepsilon_0 \nu'_0) 2 n_0\right| < \quad \text{or} \quad > \varepsilon_0 \nu_0. \tag{32}$$

Thus, the growth rates decrease or increase with the increase in kinematic viscosity and increase or decrease with the increase in kinematic viscoelasticity.

Also, it is evident from (29) that if

$$\nu'_0 < \frac{k_{10}}{\varepsilon_0} \quad \text{and} \quad \nu'_0 < \frac{\nu_0}{n_0}, \tag{33}$$

i.e. if

$$\nu'_0 < \min\left(\frac{k_{10}}{\varepsilon_0}, \frac{\nu_0}{n_0}\right), \tag{34}$$

then dn_0 / dk_{10} is always positive. The growth rates, therefore, increase with the increase in medium permeability. Otherwise, growth rates increase with the increase in medium permeability if

$$\frac{k_{10}}{\varepsilon_0} < \nu'_0 < \frac{\nu_0}{n_0} \quad \text{and} \quad \left|(k_{10} - \varepsilon_0 \nu'_0) 2 n_0\right| < \varepsilon_0 \nu_0 \tag{35}$$

or

$$\nu'_0 > \max\left(\frac{k_{10}}{\varepsilon_0}, \frac{\nu_0}{n_0}\right) \quad \text{and} \quad \left|(k_{10} - \varepsilon_0 \nu'_0) 2 n_0\right| > \varepsilon_0 \nu_0. \tag{36}$$

Also, it is evident from (29) that if

$$\nu'_0 < \frac{k_{10}}{\varepsilon_0} \quad \text{and} \quad \nu'_0 > \frac{\nu_0}{n_0}, \tag{37}$$

i.e. if

$$\frac{\nu_0}{n_0} < \nu_0' < \frac{k_{10}}{\varepsilon_0}, \tag{38}$$

then dn_0 / dk_{10} is always negative. The growth rates, therefore, decrease with the increase in medium permeability. Otherwise, growth rates decrease with the increase in medium permeability if

$$\nu_0' > \max\left(\frac{k_{10}}{\varepsilon_0}, \frac{\nu_0}{n_0}\right) \quad \text{and} \quad \left|(k_{10} - \varepsilon_0\nu_0')2n_0\right| < \varepsilon_0\nu_0 \tag{39}$$

or

$$\frac{k_{10}}{\varepsilon_0} < \nu_0' < \frac{\nu_0}{n_0} \quad \text{and} \quad \left|(k_{10} - \varepsilon_0\nu_0')2n_0\right| > \varepsilon_0\nu_0 . \tag{40}$$

The growth rates, thus, both increase (under certain conditions) and decrease (under different conditions) with the increase in kinematic viscosity, kinematic viscoelasticity and medium permeability.

5. EFFECT OF VARIABLE HORIZONTAL MAGNETIC FIELD

The effect of a magnetic field on the stability of such a flow is of interest in geophysics, particularly in the study of the Earth's core, where the Earth's mantle, which consists of conducting fluid, behaves like a porous medium that can become convectively unstable as a result of differential diffusion. So in this section, we consider the effect of variable magnetic field on the stability problem. Here the problem and configuration are the same as these described in sec. 2 except that the incompressible Walters' (model B′) fluid layer arranged in horizontal strata is acted on by a variable horizontal magnetic field $\mathbf{H}(H_0(z), 0, 0)$. Then the equations of motion and Maxwell's equations are

$$\frac{\rho}{\varepsilon}\left[\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{\varepsilon}(\mathbf{u}.\nabla)\mathbf{u}\right] = -\nabla p - \rho g \boldsymbol{\lambda} - \frac{1}{k_1}\left(\mu - \mu'\frac{\partial}{\partial t}\right)\mathbf{u} + \frac{\mu_e}{4\pi}(\nabla \times \mathbf{H}) \times \mathbf{H} , \tag{41}$$

$$\nabla.\mathbf{H} = 0 , \tag{42}$$

$$\varepsilon\frac{\partial \mathbf{H}}{\partial t} = (\mathbf{H}.\nabla)\mathbf{u} - (\mathbf{u}.\nabla)\mathbf{H} , \tag{43}$$

together with equations (2) and (3). Here μ_e denotes the magnetic permeability.

Let $\mathbf{h}(h_x, h_y, h_z)$ denote the perturbation of the variable horizontal magnetic field $\mathbf{H}(H_0(z), 0, 0)$. Then the linearized perturbation equations governing the motion of Walters' (model B′) fluid layer through porous medium in the presence of variable horizontal magnetic field are

$$\frac{\rho}{\varepsilon}\frac{\partial \mathbf{u}}{\partial t} = -\nabla\delta p - g\delta\rho\lambda - \frac{1}{k_1}\left(\mu - \mu'\frac{\partial}{\partial t}\right)\mathbf{u} + \frac{\mu_e}{4\pi}\left[(\nabla\times\mathbf{h})\times\mathbf{H} + (\nabla\times\mathbf{H})\times\mathbf{h}\right], \quad (44)$$

$$\nabla.\mathbf{h} = 0, \quad (45)$$

$$\varepsilon\frac{\partial \mathbf{h}}{\partial t} = (\mathbf{H}.\nabla)\mathbf{u} - (\mathbf{u}.\nabla)\mathbf{H}, \quad (46)$$

together with equations (5) and (6).

For perturbations of the form (7), equations (44)–(46) give

$$\left[\frac{n}{\varepsilon} + \frac{1}{k_1}(\nu - \nu' n)\right]\rho u = -ik_x\delta p + \frac{\mu_e}{4\pi}h_z(DH_0), \quad (47)$$

$$\left[\frac{n}{\varepsilon} + \frac{1}{k_1}(\nu - \nu' n)\right]\rho \upsilon = -ik_y\delta p + \frac{\mu_e H_0}{4\pi}(ik_x h_y - ik_y h_x), \quad (48)$$

$$\left[\frac{n}{\varepsilon} + \frac{1}{k_1}(\nu - \nu' n)\right]\rho w = -D\delta p + \frac{\mu_e H_0}{4\pi}\left(ik_x h_z - Dh_x - h_x\frac{DH_0}{H_0}\right) - g\delta\rho, \quad (49)$$

$$ik_x h_x + ik_y h_y + Dh_z = 0, \quad (50)$$

$$\varepsilon n h_x = ik_x H_0 u - wDH_0, \quad (51)$$

$$\varepsilon n h_y = ik_x H_0 \upsilon, \quad (52)$$

$$\varepsilon n h_z = ik_x H_0 w, \quad (53)$$

together with equations (11) and (12).

Equation (48), with the help of (51) and (52), becomes

$$\left[\frac{n}{\varepsilon} + \frac{1}{k_1}(\nu - \nu' n)\right]\rho \upsilon = -ik_y\delta p + \frac{\mu_e H_0}{4\pi\varepsilon n}(ik_x H_0\zeta + ik_y wDH_0), \quad (54)$$

where $\zeta = ik_x\upsilon - ik_y u$ is the z-component of vorticity.

Multiplying (47) by $-ik_x$, (54) by $-ik_y$, adding and using (11), we obtain

$$\left[\frac{n}{\varepsilon}+\frac{1}{k_1}(\nu-\nu' n)\right]\rho Dw=-k^2\delta p+\frac{\mu_e k_x k_y H_0^2}{4\pi\varepsilon n}\zeta+\frac{\mu_e k_y^2 H_0}{4\pi\varepsilon n}(DH_0)w-\frac{i\mu_e k_x}{4\pi}h_z(DH_0). \tag{55}$$

Eliminating δp from (49) and (55) and using (11), (12) and (51)–(53), we arrive at

$$\frac{n}{\varepsilon}\left[D(\rho Dw)-k^2\rho w\right]+\frac{1}{k_1}\left[D\{(\mu-\mu' n)Dw\}-k^2(\mu-\mu' n)w\right]$$

$$=-\frac{\mu_e k_x^2}{4\pi\varepsilon n}\left[D\left(H_0^2 Dw\right)-H_0^2 k^2 w\right]-\frac{gk^2}{\varepsilon n}(D\rho)w. \tag{56}$$

Let us assume

$$\rho=\rho_0 e^{\beta z},\quad \mu=\mu_0 e^{\beta z},\quad \mu'=\mu_0' e^{\beta z},\quad H_0^2(z)=H_1^2 e^{\beta z},\quad \varepsilon=\varepsilon_0 e^{\beta z},\quad k_1=k_{10}e^{\beta z}, \tag{57}$$

where ρ_0, μ_0, μ_0', H_1, ε_0, k_1 and β are constants, hence

$$\nu\left(=\frac{\mu}{\rho}=\frac{\mu_0}{\rho_0}\right),\quad \nu'\left(=\frac{\mu'}{\rho}=\frac{\mu_0'}{\rho_0}\right),\quad \text{and}\quad V_A\left(=\sqrt{\frac{\mu_e H_0^2}{4\pi\rho}}=\sqrt{\frac{\mu_e H_1^2}{4\pi\rho_0}}\right),$$

the coefficient of kinematic viscosity, the coefficient of kinematic viscoelasticity, and the Alfvén velocity, respectively, are constant everywhere.

Using the stratification of the form (57), equation (56) transforms to

$$D^2 w+\beta Dw-\frac{k^2\left[n+\frac{\varepsilon_0}{k_{10}}(\nu_0-\nu_0' n)+\frac{k_x^2 V_A^2}{n}-\frac{g\beta}{n}\right]}{\left[n+\frac{\varepsilon_0}{k_{10}}(\nu_0-\nu_0' n)+\frac{k_x^2 V_A^2}{n}\right]}w=0. \tag{58}$$

We assume that the system is confined between two rigid planes $z = 0$ and $z = d$, then vanishing of w at $z = 0$ is satisfied by the choice

$$w=A(e^{m_1 z}-e^{m_2 z}), \tag{59}$$

where

$$m_{1,2} = -\frac{\beta}{2} \pm \frac{1}{2}\left[\beta^2 + \frac{4k^2\left\{n + \frac{\varepsilon_0}{k_{10}}(v_0 - v_0' n) + \frac{k_x^2 V_A^2}{n} - \frac{g\beta}{n}\right\}}{\left\{n + \frac{\varepsilon_0}{k_{10}}(v_0 - v_0' n) + \frac{k_x^2 V_A^2}{n}\right\}}\right]^{1/2}, \tag{60}$$

while vanishing of w at $z = d$ requires

$$e^{(m_1 - m_2)d} = 1, \tag{61}$$

which implies that

$$(m_1 - m_2)d = 2is\pi, \tag{62}$$

where s is an integer.

Inserting the values of m_1 and m_2 from equation (60) into equation (62) and simplifying the latter, we obtain

$$\left[n^2\left(1 - \frac{v_0'\varepsilon_0}{k_{10}}\right) + n\left(\frac{v_0\varepsilon_0}{k_{10}}\right) + k_x^2 V_A^2\right](4k^2d^2 + \beta^2 d^2 + 4s^2\pi^2) - 4k^2d^2 g\beta = 0. \tag{63}$$

Equation (63) is the dispersion relation governing the effects of viscoelasticity, porosity, medium permeability and magnetic field on stability of stratified (exponentially varying density, viscosity, kinematic viscosity and magnetic field) viscoelastic (Walters' B′) fluids in stratified (exponentially varying medium porosity and medium permeability) porous medium.

(a) **Stable stratifications** $(\beta < 0)$. For the stable stratification $(\beta < 0)$, equation (63) does not admit any positive root of n, if (23) holds. So, the system is stable for disturbances of all wave numbers if (23) holds.

And if (24) holds, then equation (63) has one change in sign implying the existence of one positive root of n and so the system is unstable for disturbances of all wave numbers. Thus for stable stratifications $(\beta < 0)$, the system is stable or unstable, depending on

$$v_0' < \quad \text{or} \quad > \frac{k_{10}}{\varepsilon_0}.$$

(b) **Unstable stratifications** $(\beta > 0)$. For the unstable stratifications $(\beta > 0)$, equation (63) has at least one change of sign if

$$k_x^2 V_A^2 < \frac{4k^2d^2 g\beta}{(4k^2d^2 + \beta^2 d^2 + 4s^2\pi^2)}, \tag{64}$$

and if

$$\nu_0' > \frac{k_{10}}{\varepsilon_0} \quad \text{and} \quad k_x^2 V_A^2 > \frac{4k^2 d^2 g\beta}{(4k^2 d^2 + \beta^2 d^2 + 4s^2\pi^2)}, \tag{65}$$

then equation (63) has exactly one change in sign. Thus, the existence of at least one or exactly one positive root of n implies that the system is unstable for the disturbances of all wave numbers. However, the system can be completely stabilized if

$$\nu_0' < \frac{k_{10}}{\varepsilon_0} \quad \text{and} \quad k_x^2 V_A^2 > \frac{4k^2 d^2 g\beta}{(4k^2 d^2 + \beta^2 d^2 + 4s^2\pi^2)}. \tag{66}$$

Thus if $\nu_0' < k_{10}/\varepsilon_0$, then magnetic field succeeds in stabilizing wave numbers in the range

$$k^2 > \frac{g\beta}{V_A^2}\sec^2\theta - \frac{\beta^2 d^2 + 4s^2\pi^2}{4d^2}, \tag{67}$$

which were unstable in the absence of magnetic field. Here θ is the angle between k_x and k (i.e. $k_x = k\cos\theta$). But if

$$\beta > 0 \quad \text{and} \quad k_x^2 V_A^2 < \frac{4k^2 d^2 g\beta}{(4k^2 d^2 + \beta^2 d^2 + 4s^2\pi^2)}, \tag{68}$$

equation (63) has one positive root and hence the system is unstable for all wave numbers.

Let n_0 denote the positive root of (63). Then

$$\left[n_0^2\left(1 - \frac{\nu_0'\varepsilon_0}{k_{10}}\right) + n_0\left(\frac{\nu_0\varepsilon_0}{k_{10}}\right) + k_x^2 V_A^2\right](4k^2 d^2 + \beta^2 d^2 + 4s^2\pi^2) - 4k^2 d^2 g\beta = 0. \tag{69}$$

In order to study the behaviour of growth rate of unstable modes with respect to magnetic field, we examine the nature of dn_0/dV_A analytically. Equation (69) yields

$$\frac{dn_0}{dV_A} = -\frac{2k_x^2 V_A^2 k_{10}}{\{(k_{10} - \varepsilon_0\nu_0')2n_0 + \varepsilon_0\nu_0\}}. \tag{70}$$

It is evident from (70) that if (30) holds, then dn_0/dV_A is always negative. The growth rates, therefore, decrease with the increase in the magnetic field.

If (31) holds, then dn_0/dV_A is negative or positive, respectively, if

$$|(k_{10} - \varepsilon_0\nu_0')2n_0| < \quad \text{or} \quad > \varepsilon_0\nu_0. \tag{71}$$

The growth rates, therefore, decrease or increase with the increase in magnetic field.

6. EFFECT OF UNIFORM VERTICAL ROTATION

In many geophysical situations, the effect of rotation in porous medium is also important. So in this section, we consider the effect of rotation on this stability problem. Here the problem and configuration are the same as these described in sec. 2 except that the incompressible Walters' (model B′) fluid layer arranged in horizontal strata is acted on by a uniform vertical rotation $\mathbf{\Omega}(0,\ 0,\ \Omega)$. Then the equations of motion are

$$\frac{\rho}{\varepsilon}\left[\frac{\partial \mathbf{u}}{\partial t}+\frac{1}{\varepsilon}(\mathbf{u}.\nabla)\,\mathbf{u}\right]=-\nabla p-\rho g\boldsymbol{\lambda}-\frac{1}{k_1}\left(\mu-\mu'\frac{\partial}{\partial t}\right)\mathbf{u}+\frac{2\rho}{\varepsilon}(\mathbf{u}\times\mathbf{\Omega}), \tag{72}$$

together with equations (2) and (3).

Then the linearized perturbation equations governing the motion of the Walters' (model B′) fluid layer through porous medium in the presence of uniform vertical rotation are

$$\frac{\rho}{\varepsilon}\frac{\partial \mathbf{u}}{\partial t}=-\nabla\delta p-g\delta\rho\boldsymbol{\lambda}-\frac{1}{k_1}\left(\mu-\mu'\frac{\partial}{\partial t}\right)\mathbf{u}+\frac{2\rho}{\varepsilon}(\mathbf{u}\times\mathbf{\Omega}), \tag{73}$$

together with equations (5) and (6).

For perturbations of the form (7), equations (73) give

$$\left[\frac{n}{\varepsilon}+\frac{1}{k_1}(\nu-\nu'n)\right]\rho u=-ik_x\delta p+\frac{2\rho v\Omega}{\varepsilon}, \tag{74}$$

$$\left[\frac{n}{\varepsilon}+\frac{1}{k_1}(\nu-\nu'n)\right]\rho v=-ik_y\delta p-\frac{2\rho u\Omega}{\varepsilon}, \tag{75}$$

$$\left[\frac{n}{\varepsilon}+\frac{1}{k_1}(\nu-\nu'n)\right]\rho w=-D\delta p-g\delta\rho, \tag{76}$$

together with equations (11) and (12).

Multiplying (74) by $-ik_x$, (75) by $-ik_y$, adding and using (11), we obtain

$$\left[\frac{n}{\varepsilon}+\frac{1}{k_1}(\nu-\nu' n)\right]\rho Dw=-k^2\delta p-\frac{2\rho\Omega}{\varepsilon}\zeta\,, \tag{77}$$

where $\zeta = ik_x v - ik_y u$ is the z-component of vorticity.

Multiplying (74) by $-ik_y$, (75) by ik_x and adding (11), we obtain

$$\zeta=\left[\frac{2\Omega Dw}{n+\frac{\varepsilon}{k_1}(\nu-\nu' n)}\right]. \tag{78}$$

Eliminating δp from (76) and (77) and using (12) and (78), we arrive at

$$\frac{1}{\varepsilon}\left[n+\frac{4\Omega^2}{\left\{n+\frac{\varepsilon}{k_1}(\nu-\nu' n)\right\}}\right]D(\rho Dw)-\frac{k^2 n\rho w}{\varepsilon}+\frac{1}{k_1}\left[D\{(\mu-\mu' n)Dw\}-k^2(\mu-\mu' n)w\right]$$

$$+\frac{gk^2}{\varepsilon n}(D\rho)w=0\,. \tag{79}$$

Using the stratification of the form (15), equation (79) transforms to

$$D^2w+\beta Dw-\frac{k^2\left[n+\frac{\varepsilon_0}{k_{10}}(\nu_0-\nu_0' n)-\frac{g\beta}{n}\right]}{\left[n+\frac{\varepsilon_0}{k_{10}}(\nu_0-\nu_0' n)+\frac{4\Omega^2}{\left\{n+\frac{\varepsilon_0}{k_{10}}(\nu_0-\nu_0' n)\right\}}\right]}w=0\,. \tag{80}$$

We assume that the system is confined between two rigid planes $z = 0$ and $z = d$, then vanishing of w at $z = 0$ is satisfied by the choice

$$w=A(e^{m_1 z}-e^{m_2 z})\,, \tag{81}$$

where

$$m_{1,2} = -\frac{\beta}{2} \pm \frac{1}{2}\left[\beta^2 + \frac{4k^2\left\{n + \frac{\varepsilon_0}{k_{10}}(\nu_0 - \nu_0' n) - \frac{g\beta}{n}\right\}}{\left\{n + \frac{\varepsilon_0}{k_{10}}(\nu_0 - \nu_0' n) + \frac{4\Omega^2}{\left\{n + \frac{\varepsilon_0}{k_{10}}(\nu_0 - \nu_0' n)\right\}}\right\}}\right]^{1/2}, \quad (82)$$

while vanishing of w at $z = d$ requires

$$e^{(m_1 - m_2)d} = 1, \quad (83)$$

which implies that

$$(m_1 - m_2)d = 2is\pi, \quad (84)$$

where s is an integer.

Inserting the values of m_1 and m_2 from equation (82) into equation (84) and simplifying the latter, we obtain

$$\left(1 - \frac{\nu_0'\varepsilon_0}{k_{10}}\right)^2 An^3 + \left(\frac{2\nu_0\varepsilon_0}{k_{10}}\right)\left(1 - \frac{\nu_0'\varepsilon_0}{k_{10}}\right)An^2$$
$$+ \left\{\left(\frac{\nu_0\varepsilon_0}{k_{10}}\right)^2 A + 4\Omega^2 B - \left(1 - \frac{\nu_0'\varepsilon_0}{k_{10}}\right)C\right\}n - \left(\frac{\nu_0\varepsilon_0}{k_{10}}\right)C = 0, \quad (85)$$

where:

$A = \beta^2 d^2 + 4k^2 d^2 + 4s^2\pi^2$,

$B = \beta^2 d^2 + 4s^2\pi^2$,

$C = 4k^2 d^2 g\beta$.

Equation (85) is the dispersion relation governing the effects of viscoelasticity, porosity, medium permeability and rotation on stability of stratified (exponentially varying density, viscosity, viscoelasiticity) viscoelastic (Walters' B′) fluid in stratified (exponentially varying medium porosity and medium permeability) porous medium.

(a) Stable stratifications $(\beta < 0)$. For the stable stratification $(\beta < 0)$, equation (85) does not admit any positive root of n, if $\nu_0' < k_{10}/\varepsilon_0$. So, the system is stable for disturbances of all wave numbers if (23) holds. And if $\nu_0' > k_{10}/\varepsilon_0$, then equation (85) has at least one change in sign implying the existence of at least one positive root of n and so the system is unstable for disturbances of all wave numbers.

Thus for stable stratifications $(\beta < 0)$, the system is stable or unstable, depending on whether $\nu_0' <$ or $> k_{10}/\varepsilon_0$. From (85), it follows that the rotation does not have any qualitative effect on the nature of the stability or instability.

(b) Unstable stratifications $(\beta > 0)$. For the unstable stratifications $(\beta > 0)$, equation (85) has one change in sign implying the existence of one positive root of n, and so the system is unstable for disturbances of all wave numbers.

Let n_0 denote the positive root of (85). Then

$$\left(1-\frac{\nu_0'\varepsilon_0}{k_{10}}\right)^2 An_0^3 + \left(\frac{2\nu_0\varepsilon_0}{k_{10}}\right)\left(1-\frac{\nu_0'\varepsilon_0}{k_{10}}\right)An_0^2$$

$$+\left\{\left(\frac{\nu_0\varepsilon_0}{k_{10}}\right)^2 A + 4\Omega^2 B - \left(1-\frac{\nu_0'\varepsilon_0}{k_{10}}\right)C\right\}n_0 - \left(\frac{\nu_0\varepsilon_0}{k_{10}}\right)C = 0. \qquad (86)$$

To study the behaviour of growth rate of unstable modes with respect to rotation, we examine the nature of $dn_0/d\Omega$ analytically. Equation (86) yields

$$\frac{dn_0}{d\Omega} = -\frac{8\Omega B n_0}{3\left(1-\frac{\nu_0'\varepsilon_0}{k_{10}}\right)^2 An_0^2 + 4\left(\frac{\nu_0\varepsilon_0}{k_{10}}\right)\left(1-\frac{\nu_0'\varepsilon_0}{k_{10}}\right)An_0 + \left\{\left(\frac{\nu_0\varepsilon_0}{k_{10}}\right)^2 A + 4\Omega^2 B - \left(1-\frac{\nu_0'\varepsilon_0}{k_{10}}\right)C\right\}}. \qquad (87)$$

It is evident from (87) that if (30) holds, $dn_0/d\Omega$ is negative or positive, respectively, if

$$\left|4\Omega^2 B - \left(1-\frac{\nu_0'\varepsilon_0}{k_{10}}\right)C\right| < \text{ or } > \left[3\left(1-\frac{\nu_0'\varepsilon_0}{k_{10}}\right)^2 An_0^2 + 4\left(\frac{\nu_0\varepsilon_0}{k_{10}}\right)\left(1-\frac{\nu_0'\varepsilon_0}{k_{10}}\right)An_0 + \left(\frac{\nu_0\varepsilon_0}{k_{10}}\right)^2 A\right]. \qquad (88)$$

The growth rates, therefore, decrease or increase with the increase in the rotation.

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